

# RANDOM WALK ON A SURFACE GROUP: BEHAVIOR OF THE GREEN'S FUNCTION AT THE SPECTRAL RADIUS

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**ABSTRACT.** It is proved that the Green's function of the simple random walk on a surface group of large genus decays exponentially in distance at the (inverse) spectral radius. It is also shown that Ancona's inequalities extend to the spectral radius  $R$ , and therefore that the Martin boundary for  $R$ -potentials coincides with the natural geometric boundary  $S^1$ . This implies that the Green's function obeys a power law with exponent  $1/2$  at the spectral radius.

## 1. INTRODUCTION

**1.1. Green's function and transition probabilities.** A countable group  $\Gamma$  is *hyperbolic* (alternatively, *word-hyperbolic*) if for some — and therefore every — finite symmetric set of generators  $A$  the Cayley graph  $G^\Gamma$  relative to the generating set  $A$  has the *thin triangle property*. See [9] and [12] for background on hyperbolic group theory. Every free group is hyperbolic; more generally, so is every finitely generated Fuchsian group; and so is the fundamental group of any compact, negatively curved manifold. If  $\Gamma$  is a *nonelementary* hyperbolic group — that is,  $\Gamma$  is neither finite nor does it contain  $\mathbb{Z}$  as a finite-index subgroup — then it is nonamenable, and so by a theorem of Kesten [18], every symmetric, nearest neighbor random walk  $X_n$  on  $\Gamma$  (that is, a random walk whose step distribution  $p(x, y) = p(x^{-1}y) = p(y^{-1}x)$  is symmetric and has support  $A$ ) has spectral radius<sup>1</sup>  $R > 1$ . Thus, for every pair of group elements  $x, y \in \Gamma$  the *Green's function*

$$(1) \quad G_r(x, y) := \sum_{n=0}^{\infty} P^x\{X_n = y\}r^n = G_r(1, x^{-1}y)$$

has radius of convergence  $R$  (here  $P^x$  denotes the probability measure on path space that makes  $X_n$  a random walk with the specified step distribution and initial point  $X_0 = x$ ). Furthermore, (cf. [30], Th. 7.8) nonamenability also implies that the random walk is  *$R$ -transient*, that is,

$$(2) \quad G_R(x, y) < \infty \quad \forall x, y.$$

The behavior of the Green's function  $G_r(x, y)$  in a neighborhood of the spectral radius is closely tied to the asymptotic behavior of the transition probabilities  $P^x\{X_n = y\}$  for large  $n$ . It is known ([10], [25], [21], [22]) that for nearest neighbor random walks on *virtually*

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<sup>1</sup>This should really be called the *inverse* spectral radius, but to avoid cumbersome terminology I will delete the modifier and refer to  $R$  as the *spectral radius*.

*free* groups (groups that contain free subgroups of finite index, for instance  $SL_2(\mathbb{Z})$ ) with positive holding probability  $P^1\{X_1 = 1\} > 0$ , the transition probabilities obey *local limit theorems*

$$(3) \quad P^x\{X_n = y\} \sim \frac{C_{x,y}}{2\sqrt{\pi}R^n n^{3/2}},$$

and the Green's function follows a power law of exponent 1/2 at the spectral radius:

$$(4) \quad G_R(x, y) - G_r(x, y) \sim C_{x,y}\sqrt{R-r} \quad \text{as } r \rightarrow R-.$$

The exponent 3/2 in (3) corresponds to the exponent 1/2 in the power law (4); in the papers [10], [25], [21], [22] the local limit theorem (3) is deduced from (4) by standard Tauberian theorems. Corresponding formulas hold for the transition probability densities (i.e., the heat kernel) and Green's functions for Brownian motion on hyperbolic space  $\mathbb{H}^d$ . It is natural to conjecture that the local limit theorem (3) and power law (4) are valid generally for aperiodic nearest neighbor random walks on arbitrary word-hyperbolic groups. The main results of this paper will show that the power law (4) for the Green's function does in fact hold for simple nearest neighbor random walks on another large class of hyperbolic groups, the *surface groups* of large genus. Unfortunately, because the Green's function for such random walks are (most likely) *not* algebraic functions of  $r$  (as they are for random walks on virtually free groups) it is not possible to deduce the local limit theorem (3) from (4) by standard Tauberian methods.

**1.2. Ancona's inequalities.** Although the arguments of this paper show only that (4) holds for simple nearest-neighbor random walks on surface groups of high genus, most of the intermediate results require only that the random walk has the property that its Green's function satisfies a system of inequalities that assert, roughly, that the Green's function  $G_R(x, y)$  is nearly submultiplicative in the arguments  $x, y \in \Gamma$ . Ancona [2] proved that such inequalities always hold for  $r < R$ : in particular, he proved, for an arbitrary symmetric nearest neighbor random walk on a hyperbolic group, that for each  $r < R$  there is a constant  $C_r < \infty$  such that for every geodesic segment  $x_0 x_1 \cdots x_m$  in (the Cayley graph of)  $\Gamma$ ,

$$(5) \quad G_r(x_0, x_m) \leq C_r G_r(x_0, x_k) G_r(x_k, x_m) \quad \forall 1 \leq k \leq m.$$

His argument depends in an essential way on the hypothesis  $r < R$  (cf. his Condition (\*)), and they leave open the possibility that the constants  $C_r$  in the inequality (5) blow up as  $r \rightarrow R$ . For nearest-neighbor random walk on a virtually free group it can be shown, by direct calculation, that the constants  $C_r$  remain bounded as  $r \rightarrow R$ , and that the inequalities (5) remain valid at  $r = R$ . The following result asserts that the same is true for random walk on the surface group  $\Gamma_g$  of large genus  $g$ . Let  $A = A_g$  be the standard symmetric set of generators for  $\Gamma_g$ : thus,

$$(6) \quad A_g = \{a_i^\pm, b_i^\pm\}_{1 \leq i \leq g},$$

and these generators satisfy the fundamental relation

$$(7) \quad \prod_{i=1}^g a_i b_i a_i^{-1} b_i^{-1} = 1.$$

**Theorem 1.** *If the genus  $g$  is sufficiently large and the step distribution is the uniform distribution on  $A_g$ , then*

- (A) *The Green's function  $G_R(1, x)$  decays exponentially in  $|x| := d(1, x)$ ; and*
- (B) *Ancona's inequalities (5) hold for all  $r \leq R$ , with a constant  $C$  independent of  $r$ .*

**Note 2.** Here and throughout the paper  $d(x, y)$  denotes the distance between the vertices  $x$  and  $y$  in the Cayley graph  $G^\Gamma$ , equivalently, distance relative to the word metric. *Exponential decay* of the Green's function means *uniform* exponential decay in all directions, that is, there are constants  $C < \infty$  and  $\varrho < 1$  such that for all  $x, y \in \Gamma_g$ ,

$$(8) \quad G_R(x, y) \leq C\varrho^{d(x, y)}.$$

A very simple argument, given in section 2.6 below, shows that for random walk on any nonamenable group  $G_R(1, x) \rightarrow 0$  as  $|x| = d(1, x) \rightarrow \infty$ . Given this, it is routine to show that exponential decay of the Green's function follows from Ancona's inequalities. Nevertheless, an independent — and simpler — proof of exponential decay is given in section 4.5.

**Note 3.** Theorem 1 (A) is a discrete analogue of the main result of Hamenstaedt [13] concerning the Green's function of the Laplacean on the universal cover of a compact negatively curved manifold. Unfortunately, Hamenstaedt's proof appears to have a serious error.<sup>2</sup> The approach taken here is quite different than that of [13].

Assertions (A)–(B) of Theorem 1 are proved in section 4 below. The argument leans on both the *planarity* of the Cayley graph  $G^\Gamma$  of a surface group and the *large isoperimetric constant* of  $\Gamma_g$  for large genus. In addition, it requires certain *a priori* estimates on the Green's function, established in section 3, specifically (see Proposition 20), that

$$(9) \quad \lim_{g \rightarrow \infty} \sup_{x \neq 1} G_R(1, x) = 0.$$

For these, the fact that the step distribution is uniform on the generating set  $A_g$  is used, together with a bound for the spectral radius  $R = R_g$  of the simple random walk on  $\Gamma_g$  due to Zuk [31] (see also Bartholdi *et al* [5] and Nagnibeda [24]):

$$(10) \quad R_g > \sqrt{g}.$$

It is conceivable that a suitable substitute for the estimate (9) could be established more generally, without the symmetry hypothesis on the step distribution and without appealing to Zuk's inequality on the spectral radius. If so, all of the results below concerning the asymptotic behavior of the Green's function would hold at this level of generality.

**1.3. Martin boundary.** One of Ancona's primary motivations in [2] was to show that the Martin boundary for  $r$ -potentials coincides with the *geometric boundary*  $\partial\Gamma$  of the group. Recall that to every hyperbolic group is attached a natural *geometric (Gromov) boundary* ([15]); for a co-compact Fuchsian group, this coincides with the circle  $S^1$  at infinity of the hyperbolic plane. It is natural to ask how the geometric boundary is related to the Martin and Poisson boundaries of random walks on  $\Gamma$ . For random walks on co-compact Fuchsian groups, Series [28] showed that the Martin boundary and the geometric boundary coincide. Ancona proved, using the inequalities (5), that this is true for random walk in any hyperbolic group, and that in fact the Martin boundary for  $r$ -potentials coincides with the

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<sup>2</sup>The error is in the proof of Lemma 3.1: The claim is made that a lower bound on a finite measure implies a lower bound for its Hausdorff-Billingsley dimension relative to another measure. This is false — in fact such a lower bound on measure implies an *upper* bound on its Hausdorff-Billingsley dimension.

geometric boundary for every  $r < R$ . Independently, by unrelated arguments, Kaimanovich [14] showed that the Poisson boundary also coincides with the geometric boundary.

**Corollary 4.** *For simple random walk on a surface group  $\Gamma_g$  of sufficiently large genus  $g$ , the Martin boundary for  $R$ -potentials coincides with the geometric boundary  $S^1$ .*

This follows from the Ancona inequalities, roughly by the same argument as used in [1], [2] for the Martin boundary at  $r < R$ . (See Theorems 5.1–5.2 of [2], Theorem 6 of [1], and of [3].) For fixed  $r \leq R$ , the statement that the Martin boundary for  $r$ -potentials coincides with the geometric boundary means that (1) for every geodesic ray  $y_0, y_1, y_2, \dots$  converging to  $\zeta \in \partial\Gamma$ , and every  $x \in \Gamma$ ,

$$(11) \quad \lim_{n \rightarrow \infty} \frac{G_r(x, y_n)}{G_r(1, y_n)} = K_r(x, \zeta) = K(x, \zeta)$$

exists; (2) for each  $\zeta \in \partial\Gamma$  the function  $K_\zeta(x) := K(x, \zeta)$  is minimal positive harmonic in  $x$ ; (3) for distinct points  $\zeta, \zeta' \in \partial\Gamma$  the functions  $K_\zeta$  and  $K_{\zeta'}$  are different; and (4) the topology of pointwise convergence on  $\{K_\zeta\}_{\zeta \in \partial\Gamma}$  coincides with the usual topology on  $\partial\Gamma = S^1$ .

In fact the Ancona inequalities yield explicit rates for the convergence (11), and imply that the Martin kernel  $K_r(x, \zeta)$  is *Holder* continuous in  $\zeta$  relative to any *visual metric* on  $\partial\Gamma$ . A visual metric  $d_{\text{vis}}$  on  $\partial\Gamma$  is a metric that induces the canonical topology on  $\partial\Gamma$  (see [16]) and has the following property: there exist constants  $C, \alpha > 0$  such that for any two distinct points  $\xi, \zeta \in \partial\Gamma$  and any bi-infinite geodesic  $\gamma$  connecting  $\zeta$  to  $\xi$ ,

$$(12) \quad C^{-1} \exp\{-\alpha d(1, \gamma)\} \leq d_{\text{vis}}(\xi, \zeta) \leq C \exp\{-\alpha d(1, \gamma)\}.$$

Here  $d$  denotes the usual word metric, and  $d(1, \gamma)$  is the distance from the root vertex 1 to the geodesic  $\gamma$  (that is, the minimal distance from 1 to a vertex on the geodesic). It is known (see [16], Th. 2.18) that there is a visual metric for every  $\alpha > 0$  sufficiently small.

**Theorem 5.** *Assume that the Ancona inequalities 5 hold for all  $r \leq R$ , with a constant  $C$  independent of  $r$ . Then there exists  $\varrho < 1$  such that for every  $r \leq R$  and every geodesic ray  $1 = y_0, y_1, y_2, \dots$  converging to a point  $\zeta \in \partial\Gamma$ ,*

$$(13) \quad \left| \frac{G_r(x, y_n)}{G_r(1, y_n)} - K_r(x, \zeta) \right| \leq C_x \varrho^n$$

for constants  $C_x < \infty$  depending on  $x \in \Gamma$  but not on  $r \leq R$ . Consequently, for each  $x \in \Gamma$  the functions  $\zeta \mapsto K_r(x, \zeta)$  are uniformly Hölder continuous (relative to a visual metric  $d_{\text{vis}}$ ) in  $\zeta$  for some exponent not depending on  $r \leq R$ , and  $r \mapsto K_r(x, \cdot)$  is continuous in the Hölder norm.

This follows from the Ancona inequalities (5) by essentially the same argument as used by Anderson & Schoen ([3], Theorems 5.1, 6.2, and Corollary 6.4) to prove the analogous theorem for the Green's function on a Cartan manifold with pinched negative curvature. Routine arguments (see [13], Lemma 2.1, also [23]) give the following corollary.

**Corollary 6.** *Assume that the Ancona inequalities 5 hold for all  $r \leq R$ , with a constant  $C$  independent of  $r$ . Then there is a continuous function  $\Lambda_r : \Gamma \times \partial\Gamma \times \partial\Gamma \rightarrow \mathbb{R}_+$  such that for each  $x \in \Gamma$  and any two distinct points  $\xi, \zeta \in \partial\Gamma$ , if  $1 = y_0, y_1, \dots$  is a geodesic ray converging to  $\xi$  and  $1 = z_0, z_1, \dots$  a geodesic ray converging to  $\zeta$ , then*

$$(14) \quad \lim_{n \rightarrow \infty} \frac{G_r(y_n, x) G_r(z_n, x)}{G_r(y_n, z_n)} = \Lambda_r(x; \xi, \zeta).$$

The function  $\Lambda_r(x; \xi, \zeta)$  vanishes when  $\xi = \zeta$ , and for each  $x \in \Gamma$  is jointly Hölder in  $\xi, \zeta$  relative to a visual metric. Furthermore, for some  $\varrho < 1$  and constants  $C_{x, \xi, \zeta} < \infty$  not depending on  $r \leq R$ ,

$$(15) \quad \left| \frac{G_r(y_n, x)G_r(z_n, x)}{G_r(y_n, z_n)} - \Lambda_r(x; \xi, \zeta) \right| \leq C_{x, \xi, \zeta} \varrho^n.$$

**1.4. Decay at infinity of the Green's function.** Neither Ancona's result nor Theorem 1 gives any information about how the uniform exponential decay rate  $\varrho$  depends on the step distribution of the random walk. In fact, as will be seen below, the Green's function  $G_r(1, x)$  decays at different rates in different directions  $x \rightarrow \partial\Gamma$ . To quantify the overall decay, consider the behavior of the Green's function over the entire sphere  $S_m$  of radius  $m$  centered at 1 in the Cayley graph  $G^\Gamma$ . Recall that if  $\Gamma$  is nonelementary and word-hyperbolic then the cardinality of the sphere  $S_m$  grows exponentially in  $m$ ; in fact (see Note 37 in section 5) there are constants  $C > 0$  and  $\zeta > 1$  such that as  $m \rightarrow \infty$ ,

$$(16) \quad |S_m| \sim C\zeta^m$$

**Theorem 7.** *Assume that  $\Gamma$  is nonelementary and word-hyperbolic, and satisfies Assumption 35 below. (This holds for all cocompact Fuchsian groups, and thus for the surface groups of genus  $g \geq 2$ .) Let  $X_n$  be a symmetric, nearest neighbor random walk on  $\Gamma$ . If Ancona's inequalities hold at the spectral radius  $R$ , then*

$$(17) \quad \lim_{m \rightarrow \infty} \sum_{x \in S_m} G_R(1, x)^2 = C > 0$$

*exists and is finite, and*

$$(18) \quad \#\{x \in \Gamma : G_R(1, x) \geq \varepsilon\} \asymp \varepsilon^{-2}$$

*as  $\varepsilon \rightarrow 0$ . (Here  $\asymp$  means that the ratio of the two sides remains bounded away from 0 and  $\infty$ .)*

The proof is carried out in section 5 below, using the fact that every word-hyperbolic group has an *automatic structure* [11]. It is likely that  $\asymp$  can be replaced by  $\sim$  in (18). Note the resemblance between relation (18) and the asymptotic formula for the number of lattice points in the ball of radius  $m$ : this is no accident, because  $\log G_R(x, y)/G_R(1, 1)$  is a metric on  $\Gamma$  quasi-isometric to the word metric (sec. 2.7 below). There is a simple heuristic argument that suggests why the sums  $\sum_{x \in S_m} G_R(1, x)^2$  should remain bounded as  $m \rightarrow \infty$ : Since the random walk is  $R$ -transient, the contribution to  $G_R(1, 1) < \infty$  from random walk paths that visit  $S_m$  and then return to 1 is bounded (by  $G_R(1, 1)$ ). For any  $x \in S_m$ , the term  $G_R(1, x)^2/G_R(1, 1)$  is the contribution to  $G_R(1, 1)$  from paths that visit  $x$  before returning to 1. Thus, if  $G_R(1, x)$  is not substantially larger than

$$\sum_{n=1}^{\infty} P^1\{X_n = x \text{ and } \tau(m) = n\} R^n,$$

where  $\tau(m)$  is the time of the first visit to  $S_m$ , then the sum in (17) should be of the same order of magnitude as the total contribution to  $G_R(1, 1) < \infty$  from random walk paths that visit  $S_m$  and then return to 1. Of course, the difficulty in making this heuristic argument rigorous is that *a priori* one does not know that paths that visit  $x$  are likely to be making their first visits to  $S_m$ ; it is Ancona's inequality (5) that ultimately fills the gap.

A simple argument shows that the sum of the Green's function on the sphere  $S_m$ , unlike the sum of its square, explodes as  $m \rightarrow \infty$ :

**Proposition 8.**

$$(19) \quad \sum_{x \in S_m} G_r(1, x) \geq r^m$$

*Proof.* Since  $X_n$  is transient, it will, with probability one, eventually visit the sphere  $S_m$ . Since the steps of the random walk are of size 1, the minimum number of steps needed to reach  $S_m$  is  $m$ . Hence,

$$\begin{aligned} \sum_{x \in S_m} G_r(1, x) &= \sum_{n=m}^{\infty} \sum_{x \in S_m} P^1\{X_n = x\} r^n \\ &\geq r^m \sum_{n=m}^{\infty} P^1\{X_n \in S_m\} \\ &\geq r^m P^1\{X_n \in S_m \text{ for some } n\} \\ &= r^m. \end{aligned}$$

□

### 1.5. Critical exponent for the Green's function.

**Theorem 9.** *Assume that  $\Gamma$  is nonelementary and word-hyperbolic, that the Cayley graph of  $\Gamma$  is planar, and that Assumption 35 below is satisfied. Let  $X_n$  be a symmetric, nearest neighbor random walk on  $\Gamma$ . If Ancona's inequalities hold at the spectral radius  $R$ , then there exist constants  $C_{x,y} > 0$  such that as  $r \rightarrow R-$ ,*

$$(20) \quad G_R(x, y) - G_r(x, y) \sim C_{x,y} \sqrt{R - r}.$$

The proof, like that of Theorem 7, uses the existence of an automatic structure and the attendant thermodynamic formalism, and also relies critically on assertion (17), which forces the value of the key thermodynamic variable. The hypothesis that the Cayley graph is planar is probably extraneous; it is used only in the proof of the technical Lemma 50 in sec. 7.2. The proof of Theorem 9 is carried out in section 7.

**1.6. Standing Conventions.** The values of constants  $C$ ,  $C_x$ , and so on may change from line to line. The symbol  $R$  is reserved for the spectral radius of the random walk. The Green's function will be denoted by  $G_r(x, y)$  throughout, but the symbol  $G$  is also used with superscript  $\Gamma$  to denote the Cayley graph of  $\Gamma$ . The symbol  $\sim$  is used in the conventional way, meaning that the ratio of the two sides approaches 1.

## 2. GREEN'S FUNCTION AND RELATED GENERATING FUNCTIONS: PRELIMINARIES

Throughout this section,  $X_n$  is a symmetric, nearest neighbor random walk on a finitely generated, nonamenable group  $\Gamma$  with (symmetric) generating set  $A$ .

**2.1. Green's function as a sum over paths.** The Green's function  $G_r(x, y)$  defined by (1) has an obvious interpretation as a sum over paths from  $x$  to  $y$ . (Note: Here and in the sequel a *path* in  $\Gamma$  is just the sequence of vertices visited by a path in the Cayley graph  $G^\Gamma$ , that is, a sequence of group elements such that any two successive elements differ by right-multiplication by a generator  $a \in A$ .) Denote by  $\mathcal{R}(x, y)$  the set of all paths  $\gamma$  from  $x$  to  $y$ , and for any such path  $\gamma = (x_0, x_1, \dots, x_m)$  define the *weight*

$$(21) \quad w_r(\gamma) := r^m \prod_{i=0}^{m-1} p(x_i, x_{i+1}).$$

Then

$$(22) \quad G_r(x, y) = \sum_{\gamma \in \mathcal{R}(x, y)} w_r(\gamma).$$

Since the step distribution  $p(a) = p(a^{-1})$  is symmetric with respect to inversion, so is the weight function  $\gamma \mapsto w_r(\gamma)$ : if  $\gamma^R$  is the reversal of the path  $\gamma$ , then  $w_r(\gamma^R) = w_r(\gamma)$ . Consequently, the Green's function is symmetric in its arguments:

$$(23) \quad G_r(x, y) = G_r(y, x).$$

Also, the weight function is multiplicative with respect to concatenation of paths, that is,  $w_r(\gamma\gamma') = w_r(\gamma)w_r(\gamma')$ . Since the step distribution  $p(a) > 0$  is strictly positive on the generating set  $A$ , it follows that the Green's function satisfies a system of *Harnack inequalities*: There exists a constant  $C < \infty$  such that for each  $0 < r \leq R$  and all group elements  $x, y, z$ ,

$$(24) \quad G_r(x, z) \leq C^{d(y, z)} G_r(x, y).$$

**2.2. First-passage generating functions.** Other useful generating functions can be obtained by summing path weights over different sets of paths. Two classes of such generating functions that will be used below are the *restricted Green's functions* and the *first-passage* generating functions (called the *balayage* by Ancona [2]) defined as follows. Fix a region  $\Omega \subset G^\Gamma$ , and for any two vertices  $x, y \in G^\Gamma$  let  $\mathcal{P}(x, y; \Omega)$  be the set of all paths from  $x$  to  $y$  that remain in the region  $\Omega$  at all except the initial and final points. Define

$$(25) \quad G_r(x, y; \Omega) = \sum_{\mathcal{P}(x, y; \Omega)} w_r(\gamma), \quad \text{and}$$

$$F_r(x, y) = G_r(x, y; G^\Gamma \setminus \{y\}).$$

Thus,  $F_r(x, y)$ , the *first-passage generating function*, is the sum over all paths from  $x$  to  $y$  that first visit  $y$  on the last step. This generating function has the alternative representation

$$(26) \quad F_r(x, y) = E^x r^{\tau(y)}$$

where  $\tau(y)$  is the time of the first visit to  $y$  by the random walk  $X_n$ , and the expectation extends only over those sample paths such that  $\tau(y) < \infty$ . Note that the restricted Green's functions  $G_r(\cdot, \cdot; \Omega)$  obey Harnack inequalities similar to (24), but with the distance  $d(y, z)$  replaced by the distance  $d_\Omega(y, z)$  in the set  $\Omega$ . Finally, since any visit to  $y$  by a path started at  $x$  must follow a *first* visit to  $y$ ,

$$(27) \quad G_r(x, y) = F_r(x, y)G_r(1, 1).$$

Therefore, since  $G_r$  is symmetric in its arguments, so is  $F_r$ .

**2.3. Renewal equation for the Green's function.** The representation (1) suggests that  $G_r(1, 1)$  can be interpreted as the expected “discounted” number of visits to the root 1 by the random walk, where the discount factor is  $r$ . Any such visit must either occur at time  $n = 0$  or after the first step, which must be to a generator  $x \in A$ . Conditioning on the first step and using the Markov property, together with the symmetry  $F_r(1, x) = F_r(x, 1)$ , yields the *renewal equation*

$$G_r(1, 1) = 1 + \sum_{x \in A} p_x r F_r(1, x) G_r(1, 1),$$

which may be rewritten in the form

$$(28) \quad G_r(1, 1) = 1 / \left( 1 - \sum_{x \in A} p_x r F_r(1, x) \right).$$

Since  $G_R(1, 1) < \infty$  (recall that the group  $\Gamma$  is nonamenable), it follows that

$$(29) \quad \sum_{x \in A} p_x R F_r(1, x) < 1.$$

**2.4. Retracing inequality.** The first-passage generating function  $F_r(1, x)$  is the sum of weights of all paths that first reach  $x$  at the last step. For  $x \in A$  this may occur in one of two ways: either the path jumps from 1 to  $x$  at its first step, or it first jumps to some  $y \neq x$  and then later finds its way to  $x$ . The latter will occur if the path returns to the root 1 from  $y$  without visiting  $x$ , and then finds its way from 1 to  $x$ . This leads to a simple bound for  $F_r(1, x)$  in terms of the *avoidance generating function*  $A_r(1; x)$  defined by

$$(30) \quad A_r(1; x) := \sum_{y \neq x} p_y R F_r(y, 1; \{x\}, \{1\}) = G_r(1, 1; \Gamma \setminus \{x\}).$$

**Lemma 10.** *The avoidance generating function satisfies  $A_R(1; x) < 1$  for every  $x \in A$ , and for every  $r \leq R$ ,*

$$(31) \quad F_r(1, x) \geq p_x R / (1 - A_r(1; x)).$$

*For simple random walk on the surface group  $\Gamma_g$ , the inequality is strict.*

*Proof.* A path  $\gamma$  that starts at the root 1 can reach  $x$  by jumping directly from 1 to  $x$ , on the first step, or by jumping from 1 to  $x$  after an arbitrary number  $n \geq 1$  of returns to 1 without first visiting  $x$ . Hence,

$$F_r(1, x) \geq p_x R \left\{ 1 + \sum_{n=1}^{\infty} A_r(1; x)^n \right\}.$$

The inequality is strict for random walk on the surface group because in this case there are positive-probability paths from 1 to  $x$  that do not end in a jump from 1 to  $x$ . Since  $F_R(1, x) < \infty$ , by (27) and (2), it must be that  $A_R(1; x) < \infty$ .  $\square$

**2.5. Renewal inequality.** The avoidance generating functions can be used to reformulate the renewal equation (28) in a way that leads to a useful upper bound for the Green's function. Recall that the renewal equation was obtained by splitting paths that return to the root 1 at the time of their *first* return. Consider a path  $\gamma$  starting at 1 that first returns to 1 only at its last step: such a path must either avoid  $x \in A$  altogether, or it must visit



$x$  before the first return to 1, and then subsequently find its way back to 1. Thus, for any generator  $x \in A$ ,

$$\begin{aligned} G_r(1, 1) &= 1 + A_r(1; x)G_r(1, 1) + F_r(1, x; \Omega \setminus \{1\})F_r(x, 1)G_r(1, 1) \\ &\leq 1 + A_r(1; x)G_r(1, 1) + F_r(x, 1)^2 G_r(1, 1). \end{aligned}$$

Solving for  $G_r(1, 1)$  gives the following *renewal inequality*:

$$(32) \quad G_r(1, 1) \leq \{1 - A_r(1; x) - F_r(1, x)^2\}^{-1}.$$

**2.6. Backscattering.** A very simple argument shows that the Green's function  $G_R(1, x)$  converges to 0 as  $|x| \rightarrow \infty$ . Observe that if  $\gamma$  is a path from 1 to  $x$ , and  $\gamma'$  a path from  $x$  to 1, then the concatenation  $\gamma\gamma'$  is a path from 1 back to 1. Furthermore, since any path from 1 to  $x$  or back must make at least  $|x|$  steps, the length of  $\gamma\gamma'$  is at least  $2|x|$ . Consequently, by symmetry,

$$(33) \quad F_R(1, x)^2 G_R(1, 1) \leq \sum_{n=2|x|}^{\infty} P^1\{X_n = 1\} R^n$$

Since  $G_R(1, 1) < \infty$ , by nonamenability of the group  $\Gamma$ , the tail-sum on the right side of inequality (33) converges to 0 as  $|x| \rightarrow \infty$ . Several variations on this argument will be used later.

**2.7. Subadditivity and the random walk metric.** The concatenation of a path from  $x$  to  $y$  with a path from  $y$  to  $z$  is, obviously, a path from  $x$  to  $z$ . Consequently, by the Markov property (or alternatively the path representation (22) and the multiplicativity of the weight function  $w_r$ ) the function  $-\log F_r(x, y)$  is *subadditive*:

**Lemma 11.** *For each  $r \leq R$  the first-passage generating functions  $F_r(x, y)$  and  $F_r(x, y; \Omega)$  are super-multiplicative, that is, for any group elements  $x, y, z$ ,*

$$(34) \quad \begin{aligned} F_r(x, z) &\geq F_r(x, y)F_r(y, z) \quad \text{and} \\ F_r(x, y; \Omega) &\geq F_r(x, y; \Omega)F_r(y, z; \Omega). \end{aligned}$$

Together with Kingman's subadditive ergodic theorem, this implies that the Green's function  $G_r(1, x)$  must decay (or grow) at a fixed exponential rate along suitably chosen trajectories. For instance, if

$$(35) \quad Y_n = \xi_1 \xi_2 \cdots \xi_n$$

where  $\xi_n$  is an ergodic Markov chain on the alphabet  $A$ , or on the set  $A^K$  of words of length  $K$ , then Kingman's theorem implies that

$$(36) \quad \lim n^{-1} \log G_r(1, Y_n) = \alpha \quad \text{a.s.}$$

where  $\alpha$  is a constant depending only on the transition probabilities of the underlying Markov chain. More generally, if  $\xi_n$  is a suitable ergodic stationary process, then (36) will hold. Super-multiplicativity of the Green's function also implies the following.

**Corollary 12.** *The function  $d_{RW}(x, y) := \log F_R(x, y)$  is a metric on  $\Gamma$ .*

*Proof.* The triangle inequality is immediate from Lemma 11, and symmetry  $d_{RW}(x, y) = d_{RW}(y, x)$  follows from the corresponding symmetry property (23) of the Green's function. Thus, to show that  $d_{RW}$  is a metric (and not merely a pseudo-metric) it suffices to show that if  $x \neq y$  then  $F_R(x, y) < 1$ . But this follows from the fact (2) that the Green's function is finite at the spectral radius, because the path representation implies that

$$G_R(x, x) \geq 1 + F_R(x, y)^2 + F_R(x, y)^4 + \cdots.$$

□

Call  $d_{RW}$  the *random walk metric*. The Harnack inequalities imply that the random walk metric  $d_{RW}$  is dominated by a constant multiple of the word metric  $d$ . In general, there is no domination in the other direction. However:

**Proposition 13.** *If the Green's function decays exponentially in  $d(x, y)$  (that is, if inequality (8) holds for all  $x, y \in \Gamma$ ), then the random walk metric  $d_{RW}$  and the word metric  $d$  on  $\Gamma$  are quasi-isometric, that is, there are constants  $0 < C_1 < C_2 < \infty$  such that for all  $x, y \in \Gamma$ ,*

$$(37) \quad C_1 d(x, y) \leq d_{RW}(x, y) \leq C_2 d(x, y).$$

*Proof.* If inequality (8) holds for all  $x, y \in \Gamma$ , then the first inequality in (37) will hold with  $C_1 = -\log \varrho$ . □

**Note 14.** Except in the simplest cases — when the Cayley graph is a tree, as for free groups and free products of cyclic groups — the random walk metric  $d_{RW}$  does *not* extend from  $\Gamma$  to a metric on the full Cayley graph  $G^\Gamma$ . To see this, observe that if it *did* extend, then the resulting metric space  $(G^\Gamma, d_{RW})$  would be path-connected, and therefore would have the Hopf-Rinow property: any two points would be connected by a geodesic segment. But this would imply that for vertices  $x, z \in \Gamma$  such that  $d(x, z) \geq 2$  there would be a  $d_{RW}$ -geodesic segment from  $x$  to  $z$ , and such a geodesic would necessarily pass through a point  $y$  such that  $d(x, y) = 1$ . For any such triple  $x, y, z \in \Gamma$  it would then necessarily be the case that

$$F_R(x, z) = F_R(x, y)F_R(y, z).$$

This is possible only when every random walk path from  $x$  to  $z$  must pass through  $y$  — in particular, when the Cayley graph of  $\Gamma$  is a tree.

**2.8. Green's function and branching random walks.** There is a simple interpretation of the Green's function  $G_r(x, y)$  in terms of the occupation statistics of *branching random walks*. A branching random walk is built using a probability distribution  $\mathcal{Q} = \{q_k\}_{k \geq 0}$  on the nonnegative integers, called the *offspring distribution*, together with the step distribution  $\mathcal{P} := \{p(x, y) = p(x^{-1}y)\}_{x, y \in \Gamma}$  of the underlying random walk, according to the following rules: At each time  $n \geq 0$ , each particle fissions and then dies, creating a random number of offspring with distribution  $\mathcal{Q}$ ; the offspring counts for different particles are mutually independent. Each offspring particle then moves from the location of its parent by making a random jump according to the step distribution  $p(x, y)$ ; the jumps are once again mutually independent. Consider the initial condition which places a single particle at site  $x \in \Gamma$ , and denote the corresponding probability measure on population evolutions by  $Q^x$ .

**Proposition 15.** *Under  $Q^x$ , the total number of particles in generation  $n$  evolves as a Galton-Watson process with offspring distribution  $\mathcal{Q}$ . If the offspring distribution has mean  $r \leq R$ , then under  $Q^x$  the expected number of particles at location  $y$  at time  $n$  is  $r^n P^x\{X_n = y\}$ , where under  $P^x$  the process  $X_n$  is an ordinary random walk with step distribution  $\mathcal{P}$ . Therefore,  $G_r(x, y)$  is the mean total number of particle visits to location  $y$ .*

*Proof.* The first assertion follows easily from the definition of a Galton-Watson process – see [4] for the definition and basic theory. The second is easily proved by induction on  $n$ . The third then follows from the formula (1) for the Green's function.  $\square$

There are similar interpretations of the restricted Green's function  $G_r(x, y; \Omega)$  and the first-passage generating function  $F_r(x, y)$ . Suppose that particles of the branching random walk are allowed to reproduce only in the region  $\Omega$ ; then  $G_r(x, y; \Omega)$  is the mean number of particle visits to  $y$  in this modified branching random walk.

### 3. A PRIORI ESTIMATES FOR THE SURFACE GROUP

**3.1. Symmetries of simple random walk on  $\Gamma_g$ .** Recall that the generating set  $A_g$  of the surface group  $\Gamma_g$  consists of  $2g$  letters  $a_i, b_i$  and their inverses, which are subject to the relation  $\prod [a_i, b_i] = 1$ . This fundamental relation implies others, including

$$(38) \quad \prod_{i=0}^{g-1} [b_{g-i}, a_{g-i}] = 1 \quad \text{and}$$

$$(39) \quad \prod_{i=k+1}^g [a_i, b_i] \prod_{i=1}^k [a_i^{-1}, b_i^{-1}] = 1.$$

Since each of these has the same form as the fundamental relation, each leads to an automorphism of the group  $\Gamma_g$ : relation (38) implies that the bijection

$$\begin{aligned} a_i^{\pm 1} &\mapsto b_{g-i}^{\pm 1}, \\ b_i^{\pm 1} &\mapsto a_{g-i}^{\pm 1} \end{aligned}$$

extends to an automorphism, and similarly relation (39) implies that the mapping

$$\begin{aligned} a_i^{\pm 1} &\mapsto a_{i+1}^{\pm 1} \quad 1 \leq i \leq g-1, \\ b_i^{\pm 1} &\mapsto b_{i+1}^{\pm 1} \quad 1 \leq i \leq g-1, \\ a_g^{\pm 1} &\mapsto a_1^{\mp 1}, \\ b_g^{\pm 1} &\mapsto b_1^{\mp 1} \end{aligned}$$

extends to an automorphism. Clearly, each of these automorphisms preserves the uniform distribution on  $A_g$ , and so it must also fix each of the generating functions  $G_r(1, x)$ ,  $F_r(1, x)$ , and  $A_r(1; x)$ . This implies

**Corollary 16.** *For simple random walk on  $\Gamma_g$ ,*

$$(40) \quad \begin{aligned} A_r(1; x) &= A_r(1; y) := A_r \quad \forall x, y \in A_g, \\ F_r(1, x) &= F_r(1, y) := F_r \quad \forall x, y \in A_g, \quad \text{and} \\ G_r(1, x) &= G_r(1, y) := G_r \quad \forall x, y \in A_g. \end{aligned}$$

Consequently, by the renewal equation,  $G_r = 1/(1 - rF_r)$ . Since  $G_R < \infty$ , this implies that for the simple random walk on  $\Gamma_g$  the first-passage generating functions  $F_r(1, x)$  for  $x \in A_g$  are bounded by  $1/R$ :

$$(41) \quad F_r \leq F_R < 1/R.$$

**3.2. Large genus asymptotics for  $G_R(1, 1)$ .** For simple random walk on  $\Gamma_g$ , the symmetry relations (40) and the inequalities (31), (32), and (10) can be combined to give upper bounds for the Green's function. Asymptotically, these take the following form:

**Proposition 17.**  $\lim_{g \rightarrow \infty} G_R(1, 1) = 2$ .

*Proof.* Proposition 18 below implies that  $\liminf \geq 2$ , so it suffices to prove the reverse inequality  $\limsup \leq 2$ . For notational convenience, write  $G = G_R(1, 1)$ ,  $F = F_R(1, x)$ , and  $A = A_R(1; x)$ ; by Corollary 16, the latter two quantities do not depend on the generator  $x$ . As noted above, the renewal equation implies that

$$RF = 1 - 1/G,$$

and by (41) above,  $F < 1/R$ . By Zuk's inequality (10), the spectral radius  $R = R_g$  is at least  $\sqrt{g}$ , so it follows that  $F < 1/\sqrt{g}$ , which is asymptotically negligible as the genus  $g \rightarrow \infty$ . On the other hand, the retracing inequality (31), together with Zuk's inequality, gives

$$1 - 1/G = RF > R^2/(4g(1 - A)) > 1/(4(1 - A)).$$

This implies that  $A < 1$ . In the other direction, the renewal inequality (32) implies that

$$1/G \geq (1 - A - F^2).$$

Combining the last two inequalities yields

$$1/4(1 - A) < A + F^2$$

Since  $F \rightarrow 0$  as  $g \rightarrow \infty$ , it follows that  $\liminf_{g \rightarrow \infty} A \geq 1/2$ . Finally, using once again the renewal inequality (32) and the fact that  $F$  is asymptotically negligible as  $g \rightarrow \infty$ ,

$$(42) \quad \limsup_{g \rightarrow \infty} G_R(1, 1) \leq 2.$$

□

**3.3. The covering random walk.** The simple random walk  $X_n$  on the surface group  $\Gamma_g$  can be lifted in an obvious way to a simple random walk  $\tilde{X}_n$ , called the *covering random walk*, on the free group  $\mathcal{F}_{2g}$  on  $2g$  generators. Clearly, on the event  $\tilde{X}_{2n} = 1$  that the lifted walk returns to the root at time  $2n$ , it must be the case that the projection  $X_{2n} = 1$  in  $\Gamma_g$ . Thus, the return probabilities for  $X_{2n}$  are bounded below by those of  $\tilde{X}_{2n}$ , and so the spectral radius  $R$  is bounded above by the spectral radius  $\tilde{R}$  of the covering random walk. The return probabilities of the covering random walk are easy to estimate. Each step of  $\tilde{X}_n$  either increases or decreases the distance from the root 1 by 1; the probability that the distance increases is  $(4g - 1)/4g$ , unless the walker is at the root, in which case the probability that the distance increases is 1. Consequently, the probability that  $\tilde{X}_{2n} = 1$  can be estimated from below by counting up/down paths of length  $2n$  in the nonnegative integers that begin and end at 0. It is well known that the number of such paths is the  $n$ th *Catalan number*  $\kappa_n$ . Thus,

$$(43) \quad P^1\{X_{2n} = 1\} \geq P^1\{\tilde{X}_{2n} = 1\} \geq \kappa_n \left( \frac{4g - 1}{4g} \right)^n \left( \frac{1}{4g} \right)^n.$$

The generating function of the Catalan numbers is (see )

$$\sum_{n=0}^{\infty} \kappa_n z^n = \frac{1 - \sqrt{1 - 4z}}{2z}.$$

The smallest positive singularity is at  $z = 1/4$ , and the value of the sum at this argument is 2. Since the spectral radius satisfies  $R_g^2/4g \rightarrow 1/4$  as  $g \rightarrow \infty$ , by Zuk's inequality and results of Kesten [19], the inequality (43) has the following consequence.

**Proposition 18.** *For every  $\varepsilon > 0$  there exist  $g(\varepsilon) < \infty$  and  $m(\varepsilon) < \infty$  such that if  $g \geq g(\varepsilon)$  then*

$$(44) \quad \sum_{n=0}^{m(\varepsilon)} P^1\{X_{2n} = 1\} R^{2n} \geq 2 - \varepsilon.$$

The Green's function and first-passage generating functions for the covering random walk can be exhibited in closed form, using the renewal equation and a retracing identity. The key is that the Cayley graph of the free group is the infinite homogeneous tree  $\mathbb{T}_{4g}$  of degree  $4g$ . Since there are no cycles, for any two distinct vertices  $x, y \in \Gamma_g$  there is only one self-avoiding path (and hence only one geodesic segment) from  $x$  to  $y$ ; therefore, if  $x = x_1 x_2 \cdots x_m$  is the word representation of  $x$  then

$$(45) \quad \tilde{F}_r(1, x) = \prod_{i=1}^m \tilde{F}_r(1, x_i) = \tilde{F}_r^{|x|},$$

the last because symmetry forces  $\tilde{F}_r(1, y) = \tilde{F}_r$  to have a common value for all generators  $y$ . Now fix a generator  $x \in A_g$ , and consider the first-passage generating function  $\tilde{F}_r(1, x) = \tilde{F}_r$ : Since  $\mathbb{T}_{4g}$  has no cycles, any random walk path from 1 to  $x$  must either jump directly from 1 to  $x$ , or must first jump to a generator  $y \neq x$ , then return to 1, and then eventually find its way to  $x$ . Consequently, with  $p = p_g = 1 - q = 1/4g$ ,

$$\tilde{F}_r = pr + qr \tilde{F}_r^2,$$

from which it follows that

$$(46) \quad \begin{aligned} \tilde{F}_r &= \frac{1 - \sqrt{1 - 4pqr^2}}{2qr}, \\ \tilde{G}_r &= \frac{2q}{2q - 1 - \sqrt{1 - 4pqr^2}}, \quad \text{and} \\ \tilde{R}^2 &= \frac{1}{4pq} = \frac{4g^2}{4g - 1}. \end{aligned}$$

**3.4. Uniform bounds on the Green's function.** Recall from section 2.6 that the first-passage generating function  $F_R(1, x)$  is bounded by the tail-sums of the Green's function  $G_R(1, 1)$ : in particular,

$$(47) \quad F_R(1, x)^2 \leq F_R(1, x)^2 G_R(1, 1) \leq \sum_{n=2|x|}^{\infty} P^1\{X_n = 1\} R^n.$$

Propositions 17–18 imply that, for large genus  $g$ , these tail-sums can be made uniformly small by taking  $|x|$  sufficiently large. In fact, the first-passage generating functions can be bounded away from 1 uniformly in  $x \in \Gamma_g \setminus \{1\}$  provided the genus is sufficiently large:

**Proposition 19.** *For any  $\alpha > 3/4$  there exists  $g_\alpha < \infty$  so that*

$$(48) \quad \sup_{g \geq g_\alpha} \sup_{x \neq 1} F_R(1, x) < \sqrt{\alpha}.$$

*Proof.* By Proposition 17,  $G_R(1, 1)$  is close to 2 for large genus  $g$ . On the other hand, by inequality (43),  $P^1\{X_2 = 1\}R^2 \geq (1 - 1/4g)(R^2/4g)$ , and by taking  $g$  large this can be made arbitrarily close to  $1/4$ . Hence, by taking  $g \geq g_*$  with  $g_*$  large,

$$\sum_{n=3}^{\infty} P^1\{X_n = 1\}R^n \leq \alpha$$

where  $\alpha$  can be taken arbitrarily close to  $3/4$  by letting  $g_* \rightarrow \infty$ . Inequality (47) now implies that  $F_R(1, x)^2 \leq \alpha$  for all  $|x| \geq 2$  and all  $g \geq g_*$ . But for  $|x| = 1$ , the symmetry relations of Corollary 16 and the renewal equation (28) imply that  $F_R(1, x) < 1/R$ , which tends to 0 as  $g \rightarrow \infty$ .  $\square$

A more sophisticated version of this argument shows

**Proposition 20.**

$$(49) \quad \lim_{g \rightarrow \infty} \sup_{x \neq 1} F_R(1, x) = 0.$$

*Proof.* First, consider a vertex  $x \in \Gamma_g$  at distance  $\geq 2g$  from the root 1: By inequality (47),  $F_R(1, x)^2$  is bounded by the tail-sum  $\sum_{n \geq 4g} P\{X_n = 1\}R^n$ , which by Propositions 17–18 converges to zero as  $g \rightarrow \infty$ . Thus, it remains only to show that  $F_R(1, x) \rightarrow 0$  as  $g \rightarrow \infty$  uniformly for vertices  $x$  at distance  $< 2g$  from the root.

Fix a vertex  $x \in \Gamma_g$  such that  $|x| < 2g$ , and consider a path  $\gamma$  from 1 to  $x$ . If  $\gamma$  is of length  $< 2g$  then it has no nontrivial cycles, because the fundamental relation (7) has length  $4g$ . Consequently, it lifts to a path  $\tilde{\gamma}$  in the free group  $\mathcal{F}_{2g}$  from 1 to the unique covering point  $\tilde{x}$  of  $x$  at distance  $< 2g$  from the group identity 1 in  $\mathcal{F}_{2g}$ . Since  $R \leq \tilde{R}$ , it follows that

$$\sum_{\gamma: |\gamma| < 2g} w_R(\gamma) \leq \tilde{F}_R(1, \tilde{x}) \leq \tilde{F}_{\tilde{R}}(1, \tilde{x}) = \tilde{F}_{\tilde{R}}^{|x|}$$

where the sum is over all paths from 1 to  $x$  of length  $< 2g$ . By (46), this converges to 0 as  $g \rightarrow \infty$ . On the other hand, since the concatenation of a path  $\gamma$  from 1 to  $x$  of length  $\geq 2g$  with a path  $\gamma'$  of length  $\geq 2g$  from  $x$  to 1 is a path from 1 to 1 of length  $\geq 4g$ ,

$$\left( \sum_{\gamma: |\gamma| \geq 2g} w_R(\gamma) \right)^2 \leq \sum_{n=4g}^{\infty} P^1\{X_n = 1\}R^n;$$

here the sum is over all paths from 1 to  $x$  of length  $\geq 2g$ . By Propositions 18–19, the tail-sum on the right side converges to zero as  $g \rightarrow \infty$ .  $\square$

#### 4. THE WALKABOUT ARGUMENT

According to Australian tradition (see the film *Walkabout* directed by Nicolas Roeg), at adolescence an Aborigine male embarks on a “walkabout” for six months in the outback, tracing the path of his tribal ancestors, surviving by hunting and trapping. From the viewpoint of a random walker, the hyperbolic plane is a vast outback; failure to follow, or nearly follow, the geodesic path from one point to another necessitates walkabouts whose extents grow *exponentially* with the deviation from the geodesic path. It is this that accounts for Ancona’s inequality. The following arguments make this precise.

**4.1. Free subgroups and embedded trees.** Recall that the surface group  $\Gamma_g$  in its standard presentation has  $2g$  generators which, together with their inverses, satisfy the relation (7). Denote by  $\mathcal{F}_A^+$  and  $\mathcal{F}_A^-$  the sub-semigroups of  $\Gamma_g$  generated by  $\{a_i\}_{i \leq g}$  and  $\{a_i^{-1}\}_{i \leq g}$ , respectively, and define  $\mathcal{F}_B^\pm$  similarly.

**Proposition 21.** *The image of each of the semigroups  $\mathcal{F}_A^\pm$  and  $\mathcal{F}_B^\pm$  in the Cayley graph is a rooted tree of outdegree  $g$ . Every self-avoiding path in (the image of) any one of these semigroups is a geodesic in the Cayley graph.*

*Proof.* This is an elementary consequence of Dehn's algorithm (cf. [29]). Consider a self-avoiding path  $\alpha = a_{(1)}a_{i(2)} \cdots a_{i(m)}$  in  $\mathcal{F}_A^+$ . If this were not a geodesic segment, then there would exist a geodesic path  $\beta = x_n^{-1}x_{n-1}^{-1} \cdots x_1^{-1}$ , with  $n < m$ , such that

$$a_{(1)}a_{i(2)} \cdots a_{i(m)}x_1x_2 \cdots x_n = 1.$$

According to Dehn's algorithm, either  $a_{i(m)} = x_1^{-1}$ , or there must exist a block of between  $2g + 1$  and  $4g$  consecutive letters that can be shortened by using (a cyclic rewriting of) the fundamental relation (7). Since  $\beta$  is geodesic, this block must include the last letter of  $\alpha$ ; and because  $a$ 's and  $b$ 's alternate in the fundamental relation, it must actually begin with the last letter  $a_{i(m)}$ , and therefore include at least the first  $2g$  letters  $x_1, \dots, x_{2g}$ . Hence, the Dehn shortening results in

$$a_{(1)}a_{i(2)} \cdots a_{i(m-1)}y_1y_2 \cdots y_k = 1,$$

where  $k < m - 1$ . Therefore, by induction on  $m$ , there exists  $r \geq m - n \geq 1$  such that

$$a_{(1)}a_{i(2)} \cdots a_{i(r)} = 1.$$

But this is impossible, by Dehn. This proves that every self-avoiding path in  $\mathcal{F}_A^+$  beginning at the root 1 is geodesic, and it follows by homogeneity that every self-avoiding path in  $\mathcal{F}_A^+$  is also geodesic. Finally, this implies that the image of  $\mathcal{F}_A^+$  in the Cayley graph is a tree.  $\square$

**Note 22.** The presence of large free semigroups is a general property of word-hyperbolic groups (see for example [12], Th. 5.3.E), and for this reason it may well be possible to generalize the arguments below. The primary obstacle to generalization seems to be in obtaining suitable *a priori* estimates on the first-passage generating functions to use in conjunction with Lemma 23 below.

**4.2. Crossing a tree.** Say that a path  $\gamma$  in the Cayley graph  $G^\Gamma$  *crosses* a rooted subtree  $T$  of degree  $d$  if either it visits the root of the tree, at which time it terminates, or if it crosses each of the  $d$  subtrees  $T_i$  of  $T$  attached to the root. (In the latter case, the path must terminate at the root of the last subtree it crosses.) Observe that if  $G^\Gamma$  is planar, as when  $\Gamma = \Gamma_g$  is a surface group, this definition of crossing accords with the usual topological notion of a crossing. For a vertex  $x \in \Gamma$ , let  $\mathcal{P}(x; T)$  be the set of all paths starting at  $x$  that cross  $T$ . Let  $\mathcal{P}^m(x; T)$  be the set of all paths in  $\mathcal{P}(x; T)$  of length  $\leq m$ . Define

$$H_r(x; T) := \sum_{\gamma \in \mathcal{P}(x; T)} w_r(\gamma) \quad \text{and} \\ H_r^m(x; T) := \sum_{\gamma \in \mathcal{P}^m(x; T)} w_r(\gamma).$$

(Recall that  $w_r(\gamma)$  is the  $r$ -weight of the path  $\gamma$ , defined by (21)). The following result is the essence of the walkabout argument.

**Lemma 23.** *Suppose that  $F_r(1, x) \leq \beta$  for every  $x \neq 1$  and some constant  $\beta < 1$ . Then for every rooted subtree  $T \subset G^\Gamma$  of degree  $d \geq 2$  and every vertex  $x \notin T$ ,*

$$(50) \quad H_r(x; T) \leq \beta + \beta^d / (1 - \beta^d).$$

*Proof.* It suffices to show that the inequality holds with  $H_r(x; T)$  replaced by  $H_r^m(x; T)$  for any  $m \geq 1$ . Now the generating function  $H^m(x; T)$  is a sum over paths of length  $\leq m$ . In order that such a path  $\gamma$  starting at  $x$  crosses  $T$ , it must either visit the root of  $T$ , or it must cross each of the  $d$  offshoot tree  $T_i$ . Since these are pairwise disjoint, a path  $\gamma$  that crosses every  $T_i$  can be decomposed as  $\gamma = \gamma_1 \gamma_2 \cdots \gamma_d$ , where  $\gamma_1$  starts at  $x$  and crosses  $T_1$ , and  $\gamma_{i+1}$  starts at the endpoint of  $\gamma_i$ , in  $T_i$ , and crosses  $T_{i+1}$ . Each  $\gamma_i$  must have length at least 1; hence, since their concatenation has length  $\leq m$ , each  $\gamma_i$  must have length  $\leq m - d + 1 \leq m - 1$ . Since the sum of  $w_r(\gamma)$  over all paths  $\gamma$  from  $x$  to the root of  $T$  is no larger than  $\beta$ , by hypothesis, it follows that

$$\sup_T \sup_{x \notin T} H_r^m(x; T) \leq \beta + \left( \sup_T \sup_{x \notin T} H_r^{m-1}(x; T) \right)^d$$

Therefore, since  $H^1(x; T) \leq \beta$ ,

$$\sup_T \sup_{x \notin T} H_r^m(x; T) \leq \beta + \beta^d + \beta^{2d} + \cdots.$$

□

**4.3. Exponential decay of the Green's function.** Assume now that  $\Gamma$  has a planar Cayley graph  $G^\Gamma$ , and that this is embedded quasi-isometrically in the hyperbolic plane.

**Definition 24.** Let  $\gamma = x_0 x_1 \cdots x_m$  be a geodesic segment in the Cayley graph  $G^\Gamma$ . Say that a vertex  $x_k$  on  $\gamma$  is a *barrier point* if there are disjoint rooted subtrees  $T_k, T'_k$  in the Cayley graph, both of outdegree  $d \geq 2$ , and neither intersecting  $\gamma$ , whose roots  $y_k$  and  $z_k$  are vertices neighboring  $x_k$  on opposite sides of  $\gamma$ . Call  $T_k \cup T'_k \cup \{x_k\}$  a *barrier*, and the common outdegree  $d$  the *order* of the barrier.

Note that a barrier must disconnect the hyperbolic plane in such a way that the initial and final segments  $x_0 x_1 \cdots x_{k-1}$  and  $x_{k+1} x_{k+2} \cdots x_m$  of  $\gamma$  lie in opposite components. In the arguments of [3] and [1], the region of hyperbolic space separating two *cones* plays the role of a barrier.

**Proposition 25.** *Let  $\Gamma = \Gamma_g$  be the surface group of genus  $g \geq 2$ . There is a constant  $\kappa = \kappa_g < \infty$  such that along every geodesic segment  $\gamma$  of length  $\geq \kappa n$  there are  $n$  disjoint barriers  $B_i$ , each of order  $g$ .*

**Note 26.** The value of the constant  $\kappa$  is not important in the arguments to follow. The argument below shows that  $\kappa_g = 8g$  will work.

The proof of Proposition 25 is deferred to section 4.5 below. Given the existence of barriers, the tree-crossing Lemma 23, and the *a priori* estimate on the Green's function provided by Proposition 20, the exponential decay of the Green's function at the spectral radius follows routinely:

**Theorem 27.** *If the genus  $g$  is sufficiently large, then the Green's function  $G_R(x, y)$  of simple random walk on the surface group  $\Gamma_g$ , evaluated at the spectral radius  $R = R_g$ ,*



decays exponentially in the distance  $d(x, y)$ , that is, there exist constants  $C = C_g < \infty$  and  $\varrho = \varrho_g < 1$  such that for every  $x \in \Gamma_g$ ,

$$(51) \quad G_R(1, x) \leq C\varrho^{|x|}.$$

*Proof.* By Proposition 20, for any  $\beta > 0$  there exists  $g_\beta < \infty$  so large that if  $g \geq g_\beta$  then the first-passage generating functions of the simple random walk on  $\Gamma_g$  satisfy  $F_R(1, x) < \beta$  for all  $x \neq 1$ . By Lemma 23, the tree-crossing generating functions  $H_R(x; T)$  for trees of outdegree  $g$  satisfy  $H_R(x; T) \leq \beta + \beta^g/(1 - \beta^g)$ . If  $\beta$  is sufficiently small then it follows that for any barrier  $B = T \cup T' \cup \{y\}$  of order  $g$  and any vertex  $x \notin B$ ,

$$H_R(x; T) + H_R(x; T') + F_R(x, y) < 1/2.$$

By Proposition 25, a path  $\gamma$  from 1 to  $x$  must cross  $|x|/\kappa_g$  distinct barriers. Therefore,

$$F_R(1, x) \leq 2^{-|x|/\kappa_g}.$$

□

**4.4. Action of  $\Gamma_g$  on the hyperbolic plane.** The surface group  $\Gamma = \Gamma_g$  acts by hyperbolic isometries of the hyperbolic plane  $\mathbb{H}$ . This action provides a useful description of the Cayley graph  $G^\Gamma$ , using the tessellation  $\mathcal{T} = \{x\mathcal{P}\}_{x \in \Gamma}$  of the hyperbolic plane  $\mathbb{H}$  by fundamental polygons (“tiles”)  $x\mathcal{P}$  (see, e.g., [17], chs. 3–4); for the surface group  $\Gamma_g$  the polygon  $\mathcal{P}$  can be chosen to be a regular  $4g$ -sided polygon (cf. [17], sec. 4.3, Ex. C). The tiles serve as the vertices of the Cayley graph; two tiles are adjacent if they share a side. Thus, each group generator  $a_i^\pm, b_i^\pm$  maps  $\mathcal{P}$  onto one of the  $4g$  tiles that share sides with  $\mathcal{P}$ . The sides of  $\mathcal{P}$  (more precisely, the geodesics gotten by extending the sides) can be labeled clockwise, in sequence, as

$$A_1, B_1, \bar{A}_1, \bar{B}_1, \dots, \bar{B}_g$$

in such a way that each generator  $a_i$  maps the *exterior* of the geodesic  $A_i$  onto the *interior* of  $\bar{A}_i$ , and similarly  $b_i$  maps the exterior of the geodesic  $B_i$  onto the interior of  $\bar{B}_i$ . Observe that  $4g$  tiles meet at every vertex of  $\mathcal{P}$ ; for each such vertex, the successive group elements in some cyclic rewriting of the fundamental relation (7), e.g.,

$$a_1, a_1 b_1, a_1 b_1 a_1^{-1} \dots,$$

map the polygon  $\mathcal{P}$  in sequence to the tiles arranged around the vertex. Also, the full tessellation is obtained by drawing all of the geodesics  $x A_i, x B_i, x \bar{A}_i, x \bar{B}_i$ , where  $x \in \Gamma$ : these partition  $\mathbb{H}$  into the congruent polygons  $x\mathcal{P}$ . Call the geodesics  $A_1, B_1, \dots, \bar{B}_g$  *bounding geodesics* of  $\mathcal{P}$ , and their images by isometries  $x \in \Gamma$  *bounding geodesics* of the tessellation.

That the semigroups  $\mathcal{F}_A^{\pm 1}, \mathcal{F}_B^{\pm 1}$  defined in section 4.1 are free corresponds geometrically to the following important property of the tessellation  $\mathcal{T}$ : The exteriors of two bounding geodesics of  $\mathcal{P}$  do not intersect unless the corresponding symbols are adjacent in the fundamental relation (7), e.g., the exteriors of  $A_1$  and  $B_1$  intersect, but the exteriors of  $A_1$  and  $B_2$  do not.

**Lemma 28.** *Let  $\gamma$  be a geodesic segment in the Cayley graph that begins at  $\mathcal{P}$  and on its first step jumps from  $\mathcal{P}$  to the tile  $a_i\mathcal{P}$  (respectively,  $b_i\mathcal{P}$ , or  $a_i^{-1}\mathcal{P}$ , or  $b_i^{-1}\mathcal{P}$ ). Then  $\gamma$  must remain in the halfplane exterior to  $A_i$  (respectively,  $B_i, \bar{A}_i$ , or  $\bar{B}_i$ ) on all subsequent jumps.*

*Proof.* By induction on the length of  $\gamma$ . First,  $\gamma$  cannot recross the geodesic line  $A_i$  in  $\mathbb{H}$  in its first  $2g + 1$  steps, because to do so would require that  $\gamma$  cycle through at least  $2g + 1$  tiles that meet at one of the vertices of  $\mathcal{P}$  on  $A_i$ . This would entail completing more than

half of a cyclic rewriting of the fundamental relation (7), and so  $\gamma$  would not be a geodesic segment in the Cayley graph.

Now suppose that  $|\gamma| \geq 2g + 1$ . Since  $\gamma$  cannot complete more than  $2g$  steps of a fundamental relation, it must on some step  $j \leq 2g$  jump to a tile that does not meet  $\mathcal{P}$  at a vertex. This tile must be on the other side of a bounding geodesic  $C$  that does not intersect  $A_i$  (by the observation preceding the lemma). The induction hypothesis implies that  $\gamma$  must remain thereafter in the halfplane exterior to this bounding geodesic, and therefore in the halfplane exterior to  $A_i$ .  $\square$

**4.5. Existence of barriers: Proof of Proposition 25.** Let  $\gamma$  be a geodesic segment in the Cayley graph. Since  $\gamma$  cannot make more than  $2g$  consecutive steps in a relator sequence (a cyclic rewriting of the fundamental relation), at least once in every  $4g$  steps it must jump across a bounding geodesic  $xL$  into a tile  $x\mathcal{P}$ , and then on the next step jump across a bounding geodesic  $xL'$  that does not meet  $xL$ . By Lemma 28,  $\gamma$  must remain in the halfplane exterior to  $xL'$  afterwards. Similarly, by time-reversal,  $\gamma$  must stay in the halfplane exterior to  $xL$  up to the time it enters  $x\mathcal{P}$ . Thus, the tile  $x\mathcal{P}$  segments  $\gamma$  into two parts, past and future, that live in nonoverlapping halfplanes.

**Definition 29.** If a geodesic segment  $\gamma$  in the Cayley graph  $G^\Gamma$  enters a tile  $x\mathcal{P}$  by crossing a bounding geodesic  $xL$  and exits by crossing a bounding geodesic  $xL'$  that does not intersect  $xL$ , then the tile  $x\mathcal{P}$  — or the vertex  $x \in \Gamma$  — is called a *cut point* for  $\gamma$ .

**Lemma 30.** Let  $\gamma$  be a geodesic segment in  $G^\Gamma$  from  $u$  to  $v$ . If  $x$  is a cut point for  $\gamma$ , then it is also a barrier point. Moreover, every geodesic segment from  $u$  to  $v$  passes through  $x$ .

*Proof.* Since  $\gamma$  jumps into, and then out of  $x\mathcal{P}$  across bounding geodesics  $xL$  and  $xL'$  that do not meet, the sides  $xL''$  and  $xL'''$  of  $x\mathcal{P}$  adjacent to the side  $xL'$  are distinct from  $xL$ . Denote by  $y\mathcal{P}$  and  $z\mathcal{P}$  the tiles adjacent to  $x\mathcal{P}$  across these bounding geodesic lines  $xL''$  and  $xL'''$ . For each of these tiles  $\tau$ , at least one of the four trees rooted at  $\tau$  obtained by translation of the four semigroups of Proposition 21 will lie entirely in the intersections of the halfplanes *interior* to  $xL$  and  $xL'$ , by Lemma 28. Therefore, each of the tiles  $y\mathcal{P}$  and  $z\mathcal{P}$  is the root of a tree that does not intersect  $\gamma$ . These trees, by construction, lie on opposite sides of  $\gamma$ . This proves that  $x$  is a barrier point.

Suppose now that  $\gamma'$  is another geodesic segment from  $u$  to  $v$ . If  $\gamma'$  did not pass through the tile  $x\mathcal{P}$ , then it would have to circumvent it by passing through one of the tiles  $y\mathcal{P}$  or  $z\mathcal{P}$ . To do this would require either that it complete a relation or pass through  $g$  trees. In either case, the path  $\gamma'$  could be shortened by going through  $x\mathcal{P}$ .  $\square$

To complete the proof of Proposition 25 it remains to show that the successive barriers along  $\gamma$  constructed above are pairwise disjoint. But the attached trees at the tiles  $y\mathcal{P}$  and  $z\mathcal{P}$  were chosen in such a way that each lies entirely in the intersections of the halfplanes *interior* to  $xL$  and  $xL'$ . The past and future segments of  $\gamma$  lie in the *exteriors*. Hence, at each new barrier along (say) the future segment, the attached trees will lie in halfplanes contained in these exteriors, and so will not intersect the barrier at  $x\mathcal{P}$ .  $\square$

**4.6. Ancona's inequality.** The Ancona inequalities (5) state that the major contribution to the Green's function  $G_R(x_0, x_m)$  comes from random walk paths that pass within a bounded distance of  $x_n$ . To prove this it suffices, by Lemma 30, to show that (54) holds for *cut points*  $x_m$ . The key to this is that a path from  $x_0$  to  $x_m$  that does *not* pass within distance  $n$  of  $x_m$  must cross  $g^{n-1}$  trees of outdegree  $g$ .

**Lemma 31.** *Let  $\gamma$  be a geodesic segment from  $u$  to  $v$  that passes through the root vertex 1, and suppose that vertex 1 is a cut point for  $\gamma$ . Assume that both  $u, v$  are exterior to the sphere*

$$(52) \quad S_n := \{x \in \Gamma : |x| = n\}$$

*of radius  $n$  in the Cayley graph  $G^\Gamma$  centered at 1. If  $F_R(1, x) \leq \beta$  for all vertices  $x \neq 1$ , then*

$$(53) \quad G_R(u, v; G^\Gamma \setminus S_n) \leq 2 \{\beta + \beta^g / (1 - \beta^g)\}^{g^{n-1}}.$$

*Proof.* Since both  $u, v$  are exterior to  $S_n$ , the restricted Green's function is the sum over all paths from  $u$  to  $v$  that do not enter the sphere  $S_n$  (recall definition (25)). Since 1 is a barrier point for  $\gamma$ , there are trees  $T, T'$  of outdegree  $g$  with roots adjacent to 1 on either side of  $\gamma$ . A path from  $u$  to  $v$  that does not enter  $S_n$  must cross either  $T$  or  $T'$ , and it must do so without passing within distance  $n - 1$  of the root. Thus, it must cross  $g^{n-1}$  disjoint subtrees of either  $T$  or  $T'$ . Consequently, the result follows from Lemma 31.  $\square$

**Proposition 32.** *For all sufficiently large  $g$ , there exists  $C = C_g < \infty$  such that the Green's function of the simple random walk on the surface group  $\Gamma_g$  satisfy the Ancona inequalities: In particular, for every geodesic segment  $x_0 x_1 x_2 \cdots x_m$ , every  $1 < n < m$ , and every  $1 \leq r \leq R$ ,*

$$(54) \quad G_r(x_0, x_m) \leq C G_r(x_0, x_n) G_r(x_n, x_m).$$

*Proof.* It is certainly true that for each distance  $m < \infty$  there is a constant  $\infty > C_m \geq 1$  so that (54) holds for all geodesic segments of length  $m$ , because (by homogeneity of the Cayley graph) there are only finitely many possibilities. The problem is to show that the constants  $C_m$  remain bounded as  $m \rightarrow \infty$ .

As noted above, it suffices to consider only *cut points*  $x_n$  along the geodesic segment  $\gamma$ . For ease of notation, assume that  $\gamma$  has been translated so that the cut point  $x_n = 1$  is the root vertex of the Cayley graph, and write  $u = x_0$  and  $v = x_m$  for the initial and terminal points. Assume also that  $d(u, 1) \leq m/2$ ; this can be arranged by switching the endpoints  $u, v$ , if necessary. Thus, there is a cut point  $w$  on the geodesic segment between 1 and  $v$  so that  $.7m \leq d(u, w) \leq .8$ . Let  $S = S_{\sqrt{m}}(w)$  and  $B = B_{\sqrt{m}}(w)$  be the sphere and ball, respectively, of common radius  $\sqrt{m}$ , centered at  $w$ . Any path from  $u$  to  $v$  (or any path from 1 to  $v$ ) must either pass through the ball  $B$  or not; hence

$$G_r(u, v) = G_r(u, v; B^c) + \sum_{z \in S} G_r(u, z) G_r(z, v; B^c).$$

If  $m$  is sufficiently large that  $\sqrt{m} < .1m$ , then any point  $z \in S$  must be at distance

$$.6m \leq d(u, z) \leq .9m$$

from  $u$ . Moreover, by Lemma 30, every geodesic segment from  $u$  to  $z$  must pass through 1. (This follows because 1 is a cut point for  $\gamma$ .) Similarly, since  $w$  is also a cut point, every geodesic segment from 1 to  $v$  passes through  $w$ . Consequently, for every  $z \in S$ ,

$$G_r(u, z) \leq C_{[.9m]} G_r(u, 1) G_r(1, z).$$

By Lemma 31,

$$G_r(u, v; B^c) \leq G_R(u, v; B^c) \leq 2\alpha^{g^{\sqrt{m}}}$$

where  $\alpha = \beta + \beta^g/(1 - \beta^g) < 1/2$ , provided the genus  $g$  is sufficiently large. On the other hand, the Harnack inequalities ensure that for some  $\varrho > 0$  and all  $r \geq 1$

$$\begin{aligned} G_r(u, 1) &\geq \varrho^m \quad \text{and} \\ G_r(1, v) &\geq \varrho^m \end{aligned}$$

Therefore,

$$\begin{aligned} G_r(u, v) &= G_r(u, v; B^c) + \sum_{z \in S} G_r(u, z) G_r(z, v; B^c) \\ &\leq 2\alpha^{g\sqrt{m}} + C_{[.9m]} \sum_{z \in S} G_r(u, 1) G_r(1, z) G_r(z, v; B^c) \\ &\leq 2\alpha^{g\sqrt{m}} + C_{[.9m]} G_r(u, 1) G_r(1, v) \\ &\leq (1 + 2\alpha^{g\sqrt{m}}/\varrho^{2m}) C_{[.9m]} G_r(u, 1) G_r(1, v). \end{aligned}$$

This shows that

$$C_m \leq (1 + 2\alpha^{g\sqrt{m}}/\varrho^{2m}) C_{[.9m]},$$

and it now follows routinely that the constants  $C_m$  remain bounded as  $m \rightarrow \infty$ .  $\square$

## 5. AUTOMATIC STRUCTURE AND THERMODYNAMIC FORMALISM

**5.1. Strongly Markov groups and hyperbolicity.** A finitely generated group  $\Gamma$  is said to be *strongly Markov* (fortement Markov – see [11]) if for each finite, symmetric generating set  $A$  there exists a finite directed graph  $\mathcal{A} = (V, E, s_*)$  with distinguished vertex  $s_*$  (“start”) and a labeling  $\alpha : E \rightarrow A$  of edges by generators that meets the following specifications. Let

$$(55) \quad \mathcal{P} := \{\text{finite paths in } \mathcal{A} \text{ starting at } s_*\},$$

and for each path  $\gamma = e_1 e_2 \cdots e_m \in \mathcal{P}$ , denote by

$$(56) \quad \begin{aligned} \alpha(\gamma) &= \text{path in } G^\Gamma \text{ through } 1, \alpha(e_1), \alpha(e_1)\alpha(e_2), \dots, \quad \text{and} \\ \alpha_*(\gamma) &= \text{right endpoint of } \alpha(\gamma) = \alpha(e_1)\alpha(e_2) \cdots \alpha(e_m). \end{aligned}$$

**Definition 33.** The labeled automaton  $(\mathcal{A}, \alpha)$  is a strongly Markov automatic structure for  $\Gamma$  if:

- (A) No edge  $e \in E$  ends at  $s_*$ .
- (B) Every vertex  $v \in V$  is accessible from the start state  $s_*$ .
- (C) For every path  $\gamma \in \mathcal{P}$ , the path  $\alpha(\gamma)$  is a geodesic path in  $G^\Gamma$ .
- (D) The endpoint mapping  $\alpha_* : \mathcal{P} \rightarrow \Gamma$  induced by  $\alpha$  is a bijection of  $\mathcal{P}$  onto  $\Gamma$ .

**Theorem 34.** *Every word hyperbolic group is strongly Markov.*

See [11], Ch. 9, Th. 13. The result is essentially due to Cannon [9], [7], [8], and in important special cases (cocompact Fuchsian groups) to Series [27]. Henceforth, the directed graph  $\mathcal{A} = (V, E, s_*)$  will be called the *Cannon automaton* (despite the fact that it is not the same automaton as constructed in [9]).

**5.2. Symbolic dynamics.** Call a vertex  $v \in V$  *recurrent* if there is a path in the Cannon automaton of length  $\geq 1$  that begins and ends at  $v$ ; otherwise, call it *transient*. If  $u, v \in V$  are recurrent, say that  $u$  and  $v$  *communicate* (written  $u \sim v$ ) if there are paths from  $u$  to  $v$  and from  $v$  to  $u$ . The relation  $\sim$  is an equivalence relation on the set of recurrent vertices. For cocompact Fuchsian groups (and also virtually free groups) it is known [27] that there is only one equivalence class. Henceforth we restrict attention to word-hyperbolic groups with this property:

**Assumption 35.** *There is only one recurrent class  $\mathcal{R}$ .*

Consequently, every path  $\gamma \in \mathcal{P}$  has the form  $\gamma = \tau\varrho$  where  $\tau$  is a path in the set of transient vertices whose last edge connects a transient vertex to a recurrent vertex  $v_\tau$ , and  $\varrho$  is a path in the set of recurrent vertices that starts at  $v$ . There are only finitely many possible transient prefixes  $\tau$ , since no transient vertex can be visited twice by  $\gamma$ . Thus, for sufficiently large  $m$  the sphere  $S_m$  of radius  $m$  in  $G^\Gamma$  is in one-to-one correspondence with

$$(57) \quad \bigcup_{\tau} \Sigma^{m-|\tau|}(v_\tau)$$

where

- the union  $\bigcup_{\tau}$  is over all transient prefixes  $\tau$ ,
- $v_\tau$  is the recurrent endpoint of  $\tau$ ,
- $\Sigma^k(v)$  denotes the set of paths of length  $k$  in the automaton that start at  $v$ , and
- $|\tau|$  is the length of  $\tau$ .

Let  $\Sigma$  be the set of *infinite* two-sided paths in the set of recurrent vertices, and let  $\sigma : \Sigma \rightarrow \Sigma$  be the shift operator.

**Lemma 36.** *The shift  $(\Sigma, \sigma)$  is topologically ergodic and has positive topological entropy.*

*Proof.* That  $(\Sigma, \sigma)$  is topologically ergodic follows from the assumption that all recurrent vertices communicate. Since  $\Gamma$  is nonelementary,  $|S_n|$  grows exponentially with  $n$ . Since there are only finitely many different transient prefixes  $\tau$ , the representation (57) implies that the set of paths in the set of recurrent vertices of length  $n$  must grow exponentially with  $n$ . This implies that the subshift  $(\Sigma, \sigma)$  has positive topological entropy.  $\square$

**Note 37.** The fact that  $S_m$  is in one-to-one correspondence with the set (57) implies the sharp exponential growth (16) of  $|S_m|$ , because the number of paths in the Cannon automaton of length  $m$  beginning at a recurrent vertex  $m$  is  $A^m e_v$ , where  $A$  is the incidence matrix of  $\mathcal{A}$  and  $e_v$  is the unit vector with entry 1 in slot  $v$ . The irreducibility of  $\mathcal{R}$  (Assumption 35) implies that  $A$  is a Perron-Frobenius matrix. The Perron-Frobenius eigenvalue of  $A$  is  $e^h$ , where  $h$  is the topological entropy of the shift  $(\Sigma, \sigma)$ .

**5.3. Thermodynamic formalism for the Green's function.** The representation (57) connects the asymptotic behavior of the Green's function on large spheres  $S_m$  to thermodynamic formalism under the shift  $(\Sigma, \sigma)$ . The key is Theorem 5, which implies that for each  $r \leq R$  the Green's function  $-\log G_r(1, x)$  is asymptotically a Hölder continuous *cocycle* for the dynamical system  $(\Sigma, \sigma)$ . The cocycle is constructed as follows:

Fix  $\omega \in \Sigma$ , and let

$$\omega^{(n)} = \omega_0 \omega_1 \cdots \omega_n$$

be a long segment of  $\omega$ . There exist a transient path  $\tau$  starting at  $s_*$  and a short connecting path  $\varrho$  in  $\mathcal{R}$ , neither depending on  $n$ , such that  $\tau\varrho\omega^n$  is a path in  $\mathcal{P}$ . (Note: The short connecting segment  $\varrho$  may be necessary, because the initial vertex of  $\omega$  might not be accessible

from  $s_*$  by a path that does not first visit some other recurrent vertex.) Set

$$f_n(\omega) = f_n(\omega^{(n)}) := \log \frac{G_r(\alpha_*(\tau \varrho), \alpha_*(\tau \varrho \omega^{(n)}))}{G_r(\alpha_*(\tau \varrho \omega_0), \alpha_*(\tau \varrho \omega^{(n)}))};$$

this ratio is independent of the choices of  $\tau$  and  $\varrho$ , since  $\alpha_*(\tau \varrho)^{-1} \alpha_*(\tau \varrho \omega_0)$  is determined solely by the edge  $\omega_0$ . Theorem 5 implies that the log ratio converges as  $n \rightarrow \infty$ , and that the limit function

$$(58) \quad \varphi_r(\omega) = \lim f_n(\omega) = \log \frac{K_r(\alpha_*(\tau \varrho), \alpha_*(\tau \varrho \omega))}{K_r(\alpha_*(\tau \varrho \omega_0), \alpha_*(\tau \varrho \omega))}$$

is Hölder continuous in  $\omega$  (see Bowen [6], ch. 1 for the definition of Hölder continuity on sequence space). Here  $K_r$  is the Martin kernel. Note that  $\varphi_r(\omega)$  depends only on the positive coordinates of  $\omega$  (recall that  $\Sigma$  is the set of *two-sided* infinite paths). Also, Theorem 5 implies that the extension  $\varphi_r^+ : \Sigma \cup \Sigma^* \rightarrow \mathbb{R}$  (where  $\Sigma^* = \bigcup_{n \geq 0} \Sigma^n$ ) of  $\varphi_r$  defined by

$$(59) \quad \varphi_r^+(\Omega) := \varphi_r(\Omega) \quad \text{for } \Omega \in \Sigma,$$

$$(60) \quad \varphi_r^+(\Omega) := f_n(\omega^{(n)}) \quad \text{for } \Omega \in \Sigma^n,$$

is Hölder continuous. By construction,

$$(61) \quad G_r(\alpha_*(\tau \varrho), \alpha_*(\tau \varrho \omega^{(n)})) = \exp\{S_n \varphi_r^+(\omega)\} \asymp \exp\{S_n \varphi_r(\omega)\}$$

where  $S_n \varphi := \sum_{j=0}^{n-1} \varphi \circ \sigma^j$  (in Bowen's notation [6]) and the implied constants in the relation (61) are uniform in  $\omega$ ,  $n$  and  $r \leq R$ . (Unfortunately, the notation  $S_n \varphi$  conflicts with the notation  $S_m$  for the sphere of radius  $m$  in  $\Gamma$ ; however, both notations are standard, and the meaning should be clear in the following by context.) By the Harnack inequality (24), there exists a constant  $C < \infty$  such that

$$(62) \quad \|\varphi_r\|_\infty \leq C \quad \forall r \leq R.$$

**Proposition 38.** *For each  $r < R$  and each  $1 \leq \theta < \infty$ , there exists a positive constant  $C = C(r, \theta)$ , varying continuously with  $r$  and  $\theta$ , such that uniformly for  $r \leq R$  and  $1 \leq \theta \leq K < \infty$ ,*

$$(63) \quad \sum_{x \in S_m} G_r(1, x)^\theta \sim C \exp \left\{ m \text{Pressure}(\theta \varphi_r) \right\}$$

where  $\text{Pressure}(\varphi)$  denotes the topological pressure of  $\varphi$ . For each  $\theta$ ,  $\text{Pressure}(\theta \varphi_r)$  is continuous and strictly increasing for  $r \leq R$ .

*Proof.* The asymptotic relation (63) follows by a routine argument from Ruelle's Perron-Frobenius theorem (cf. [6], ch. 1). See [20] for similar arguments. The key observation is that for each path  $\tau \varrho$  of length  $k \leq m$ , the sum of  $G_r(1, x)^\theta$  over those vertices  $x \in S_m$  with representation  $\tau \varrho \varrho'$ , where  $\varrho' \in \Sigma^{m-k}(v_\varrho)$  is a path of length  $m - k$  that starts at the endpoint  $v_\varrho$  of  $\varrho$ , is, for large  $k$ , essentially the same as  $\mathcal{L}_{\theta \varphi_r}^{m-k} 1(\varrho_*)$ , where  $\mathcal{L}_\psi$  is the Ruelle operator associated with the function  $\psi$  and  $\varrho_*$  is some infinite path in  $\Sigma$  that begins with the prefix  $\varrho$ . (The Hölder continuity of  $\varphi_r$  ensures that the error is exponentially small in  $k$ .) For fixed  $k$ , Ruelle's Perron-Frobenius theorem ensures that  $\mathcal{L}_{\theta \varphi_r}^{m-k} 1(\varrho_*)$  grows sharply exponentially in  $m$  at rate  $\text{Pressure}(\theta \varphi_r)$ . Routine use of regular perturbation theory (see [20] or [26]) shows that (63) is locally uniform in the parameters  $r, \theta$ . That the pressure is strictly increasing in  $r$  follows from (63), because for any  $x \in S_m$ ,

$$G_{r+r\varepsilon}(1, x) \geq (1 + \varepsilon)^m G_r(1, x),$$

since any path from 1 to  $x$  must have length  $\geq m$ . That it is continuous in  $r$  follows because the map  $r \mapsto \varphi_r$  is continuous relative to the Hölder metric, by Theorem 5.  $\square$

**5.4. Ergodic Theory.** For any vertex  $x \in \Gamma$  there is a *unique* geodesic segment  $L = L(1, x)$  in the Cayley graph  $G^\Gamma$  from 1 to  $x$  that corresponds to a path  $\gamma$  in the Cannon automaton  $\mathcal{A}$  (by Definition 33 and Theorem 34). The path  $\gamma$  has the form  $\tau\rho$  where  $\tau$  and  $\rho$  are paths in the set of transient and recurrent vertices of  $\mathcal{A}$ , respectively. Since a transient prefix  $\tau$  cannot visit any vertex twice, its length cannot be larger than the cardinality  $|T|$  of the set of transient vertices in the automaton  $\mathcal{A}$ . Consequently, the length  $n$  of the recurrent segment  $\rho$  is at least  $m - |T|$ . Note that  $\rho$  coincides with the segment  $\omega^n$  of a path  $\omega \in \Sigma$ .

Denote by  $\mu_{r,m}$  the probability measure on the sphere  $S_m \subset \Gamma$  with density proportional to  $G_r(1, x)^2$ . By (61), the pullback of  $\mu_{r,m}$  to  $\Sigma^*$  (via the identification  $\tau\rho \mapsto L(1, x)$ , taking only the last  $m - |T|$  letters of  $\rho$ ) coincides with the  $(m - |T|)$ -fold marginal of the *Gibbs state*  $\mu_r$  corresponding to the potential  $\varphi_r$ . (The relation (61) is effectively the definition of a Gibbs state — see [6], ch. 1.) By [6], Theorem 1.2, the Gibbs state corresponding to a Hölder continuous potential is unique. By Lemma 36, the shift  $(\Sigma, \sigma)$  is topologically mixing, and so by Proposition 1.14 of [6], the Gibbs state  $\mu_r$  is mixing, and hence ergodic, relative to  $\sigma$ . By standard arguments in regular perturbation theory [26], the Gibbs states vary continuously with  $r$  (up to  $r = R$ ) in the weak topology. This proves

**Proposition 39.** *Let  $g : \Sigma \cup \Sigma^* \rightarrow \mathbb{R}$  be any continuous function; then*

$$(64) \quad \lim_{m \rightarrow \infty} \mu_{r,m} \left\{ x \in S_m : \left| n^{-1} \sum_{j=1}^n g \circ \sigma^j(\omega^n) - \int g d\mu_r \right| > \varepsilon \right\} = 0.$$

*Moreover, the expectation  $\int g d\mu_r$  varies continuously with  $r$  for  $r \leq R$ .*

The ergodic average in (64) is expressed as an average over the orbit of the path  $\omega^n$  in the Cannon automaton, but it readily translates to an equivalent statement for ergodic averages along the geodesic segment  $L = L(1, x)$ . Each vertex  $y \in L$  disconnects  $L$  into two geodesic segments  $L^+ = L_y^+$  and  $L^- = L_y^-$ , where  $L^+$  is the segment of  $L$  from  $y$  to  $x$ , and  $L^-$  is the segment of  $L$  from  $y$  to 1. These paths determine finite words  $e^+ = e^+(y)$  and  $e^- = e^-(y)$  in the group generators  $A$  (recall that each oriented edge  $(u, v)$  of the Cayley graph is labelled by the generator  $u^{-1}v$ ). The pair  $(e^-, e^+)$  can be viewed as an element of the compact space  $\bar{A} \times \bar{A}$ , where  $\bar{A} = A^* \cup A^\infty$  is the set of all finite or infinite words in the alphabet  $A$ , endowed with the topology of coordinatewise convergence.

**Corollary 40.** *Let  $f : \bar{A} \times \bar{A} \rightarrow \mathbb{R}$  be a continuous function, and let  $g = f \circ \alpha^{-1}$  be its pullback to the space of two-sided paths in the Cannon automaton. Then*

$$(65) \quad \lim_{m \rightarrow \infty} \mu_{r,m} \left\{ x \in S_m : \left| m^{-1} \sum_{y \in L(1, x)} f(e^+(y), e^-(y)) - \int g d\mu_r \right| > \varepsilon \right\} = 0.$$

## 6. EVALUATION OF THE PRESSURE AT $r = R$

Proposition 38 implies that the sums  $\sum_{y \in S_m} G_R(1, y)^2$  grow or decay sharply exponentially at rate  $\text{Pressure}(2\varphi_R)$ . Consequently, to prove the relation (17) of Theorem 7 it suffices to prove that this rate is 0.

**Proposition 41.**  $\text{Pressure}(2\varphi_R) = 0$ .

The second assertion (18) of Theorem 7 also follows from Proposition 41, by the main result of [20]. (If it could be shown that the cocycle  $\varphi_R$  defined by (58) above is *nonlattice* in the sense of [20], then the result (18) could be strengthened from  $\asymp$  to  $\sim$ .)

The remainder of this section is devoted to the proof of Proposition 41. The first step, that  $\text{Pressure}(2\varphi_R) \leq 0$ , is a consequence of the following differential equations relating the derivative of the Green's functions  $G_r(1, x)$  to the quadratic sums  $\sum_y G_r(1, y)G_r(y, x)$ . This system of differential equations will also form the basis of the proof of Theorem 9 in section 7.

**Proposition 42.**

$$(66) \quad \boxed{\frac{d}{dr}G_r(x, y) = r^{-1} \sum_{z \in \Gamma} G_r(x, z)G_r(z, y) - r^{-1}G_r(x, y)} \quad \forall 0 \leq r < R.$$

*Proof.* This is a routine calculation based on the representation (22) of the Green's function as a sum over paths. Since all terms in the power series representation of the Green's function have nonnegative coefficients, interchange of  $d/dr$  and  $\sum_\gamma$  is permissible. If  $\gamma$  is a path from 1 to  $x$  of length  $m$ , then the derivative with respect to  $r$  of the weight  $w_r(\gamma)$  is  $mw_r(\gamma)/r$ , so  $dw_r(\gamma)/dr$  contributes one term of size  $w_r(\gamma)/r$  for each vertex visited by  $\gamma$  after its first step. This, together with the multiplicativity of  $w_r$ , yields the identity (66).  $\square$

**Corollary 43.** *For every  $r < R$ ,*

$$(67) \quad \text{Pressure}(2\varphi_r) < 0, \quad \text{and so}$$

$$(68) \quad \text{Pressure}(2\varphi_R) \leq 0.$$

*Proof.* For  $r < R$  the Green's function  $G_r(1, 1)$  is analytic in  $r$ , so its derivative must be finite. Thus, by Proposition 42, the sum  $\sum_{x \in \Gamma} G_r(1, x)^2$  is finite, because  $G_R(1, 1) < \infty$ . Proposition 38 therefore implies that  $\text{Pressure}(2\varphi_r)$  must be negative. Since  $\text{Pressure}(\varphi)$  varies continuously in  $\varphi$ , relative to the sup norm, (68) follows.  $\square$

*Proof of Proposition 41.* To complete the proof of Proposition 41, it now remains to show that  $\text{Pressure}(2\varphi_R)$  cannot be negative. In view of Proposition 38, this is equivalent to showing that  $\sum_{x \in S_m} G_R(1, x)^2$  cannot decay exponentially in  $m$ . This will be accomplished by showing that exponential decay of  $\sum_{x \in S_m} G_R(1, x)^2$  would force

$$(69) \quad G_r(1, 1) < \infty \quad \text{for some } r > R,$$

which is impossible since  $R$  is the radius of convergence of the Green's function.

To prove (69), we will use the branching random walk interpretation of the Green's function discussed in sec. 2.8.<sup>3</sup> Recall that a branching random walk on the Cayley graph  $G^\Gamma$  is specified by an offspring distribution  $\mathcal{Q}$ ; assume for definiteness that this is the Poisson distribution with mean  $r > 0$ . At each step, particles first produce offspring particles according to this distribution, independently, and then each of these particles jumps to a randomly chosen neighboring vertex. If the mean of the offspring distribution is  $r > 0$ , and if the branching random walk is initiated by a single particle at the root 1, then the mean number of particles located at vertex  $x$  at time  $n \geq 1$  is  $r^n P^1\{X_n = x\}$ . Thus, in particular,  $G_r(1, 1)$  equals the expected total number of particle visits to the root vertex 1.

<sup>3</sup>Logically this is unnecessary — the argument has an equivalent formulation in terms of weighted paths, using (22) — but the branching random walk interpretation seems more natural.



The strategy is to show that if  $\sum_{x \in S_m} G_R(1, x)^2$  decays exponentially in  $m$ , then for some  $r > R$  the branching random walk remains *subcritical*, that is, the expected total number of particle visits to 1 is finite.

Recall that the Poisson distribution with mean  $r > R$  is the convolution of Poisson distributions with means  $R$  and  $\varepsilon := r - R$ , that is, the result of adding independent random variables  $U, V$  with distributions Poisson- $R$  and Poisson- $\varepsilon$  is a random variable  $U + V$  with distribution Poisson- $r$ . Thus, each reproduction step in the branching random walk can be done by making independent draws  $U, V$  from the Poisson- $R$  and Poisson- $\varepsilon$  distributions. Use these independent draws to assign *colors*  $k = 0, 1, 2, \dots$  to the particles according to the following rules:

- (a) The ancestral particle at vertex 1 has color  $k = 0$ .
- (b) Any offspring resulting from a  $U$ -draw has the same color as its parent.
- (c) Any offspring resulting from a  $V$ -draw has color equal to 1+the color of its parent.

**Lemma 44.** *For each  $k = 0, 1, 2, \dots$ , the expected number of visits to the vertex  $y$  by particles of color  $k$  is (with the convention  $x_0 = 1$  and  $x_{k+1} = y$ )*

$$(70) \quad v_k(y) = \varepsilon^k \sum_{x_1, x_2, \dots, x_k \in \Gamma} G_R(1, x_1) \left( \prod_{i=1}^{k-1} G_R(x_i, x_{i+1}) \right) G_R(x_k, 1).$$

*Proof.* By induction on  $k$ . First, particles of color  $k = 0$  reproduce and move according to the rules of a branching random walk with offspring distribution Poisson- $R$ , so the expected number of visits to vertex  $y$  by particles of color  $k = 0$  is  $G_R(1, y)$ , by Proposition 15. This proves (70) in the case  $k = 0$ . Second, assume that the assertion is true for color  $k \geq 0$ , and consider the production of particles of color  $k + 1$ . Such particles are produced only by particles of color  $k$  or color  $k + 1$ . Call a particle a *pioneer* if its color is different from that of its parent, that is, if it results from a  $V$ -draw. Each pioneer of color  $k + 1$  engenders its own branching random walk of color  $k + 1$  descendants; the offspring distribution for this branching random walk is the Poisson- $R$  distribution. Thus, for a pioneer born at site  $z \in \Gamma$ , the expected number of visits to  $y$  by its color  $k + 1$  descendants is  $G_R(z, y)$ . Every particle of color  $k + 1$  belongs to the progeny of one and only one pioneer; consequently, the expected number of visits to  $y$  by particles of color  $k + 1$  is

$$\sum_{z \in \Gamma} u_{k+1}(z) G_R(z, 1),$$

where  $u_{k+1}(z)$  is the expected number of pioneers of color  $k + 1$  born at site  $z$  during the evolution of the branching process. But since pioneers of color  $k + 1$  must be children of parents of color  $k$ , and since for any particle the expected number of children of different color is  $\varepsilon$ , it follows that

$$u_{k+1}(z) = \varepsilon v_k(z).$$

Hence, formula (70) for  $k + 1$  follows by the induction hypothesis.  $\square$

Recall that our objective is to show that if  $\sum_{x \in S_m} G_R(1, x)^2$  decays exponentially in  $m$  then  $G_r(1, 1) < \infty$  for some  $r = R + \varepsilon > R$ . The branching random walk construction exhibits  $G_r(1, 1)$  as the expected total number of particle visits to the root vertex 1, and this is the sum over  $k \geq 0$  of the expected number  $v_k(1)$  of visits by particles of color  $k$ . Thus, to complete the proof of Proposition 41 it suffices, by Lemma 44, to show that for

some  $\varepsilon > 0$ ,

$$(71) \quad \sum_{k=0}^{\infty} \varepsilon^k \sum_{x_1, x_2, \dots, x_k \in \Gamma} G_R(1, x_1) \left( \prod_{i=1}^{k-1} G_R(x_i, x_{i+1}) \right) G_R(x_k, 1) < \infty.$$

This follows directly from the next lemma.  $\square$

**Lemma 45.** *Assume that Ancona's inequalities (5) hold at the spectral radius  $R$  with a constant  $C_R < \infty$ . If  $\sum_{x \in S_m} G_R(1, x)^2$  decays exponentially in  $m$ , then there exist constants  $\delta > 0$  and  $C, \varrho < \infty$  such that for every  $k \geq 1$ ,*

$$(72) \quad \sum_{x_1, x_2, \dots, x_k \in \Gamma} G_R(1, x_1) \left( \prod_{i=1}^{k-1} G_R(x_i, x_{i+1}) \right) (1 + \delta)^{|x_k|} G_R(x_k, 1) \leq C \varrho^k.$$

Here  $|y| = d(1, y)$  denotes the distance of  $y$  from the root 1 in the word metric.

*Proof.* Denote by  $H_k(\delta)$  the left side of (72); the strategy will be to prove by induction on  $k$  that for sufficiently small  $\delta > 0$  the ratios  $H_{k+1}(\delta)/H_k(\delta)$  remain bounded as  $k \rightarrow \infty$ . Consider first the sum  $H_1(\delta)$ : by the hypothesis that  $\sum_{x \in S_m} G_R(1, x)^2$  decays exponentially in  $m$  and the symmetry  $G_r(x, y) = G_r(y, x)$  of the Green's function,

$$(73) \quad H_1(\delta) := \sum_{x \in \Gamma} G_R(1, x)^2 (1 + \delta)^{|x|} < \infty.$$

Now consider the ratio  $H_{k+1}(\delta)/H_k(\delta)$ . Fix vertices  $x_1, x_2, \dots, x_k$ , and for an arbitrary vertex  $y = x_{k+1} \in \Gamma$ , consider its position *vis a vis* the geodesic segment  $L = L(1, x_k)$  from the root vertex 1 to the vertex  $x_k$  (cf. section 7.2). Let  $z \in L$  be the vertex on  $L$  nearest  $y$  (if there is more than one, choose arbitrarily). By the triangle inequality,

$$|y| \leq |z| + d(z, y).$$

Because the group  $\Gamma$  is word-hyperbolic, all geodesic triangles — in particular, any triangle whose sides consist of geodesic segments from  $y$  to  $z$ , from  $z$  to  $x_k$ , and from  $x_k$  to  $y$ , or any triangle whose sides consist of geodesic segments from  $y$  to  $z$ , from  $z$  to 1, and from 1 to  $y$  — are  $\Delta$ -thin, for some  $\Delta < \infty$ . Hence, any geodesic segment from  $x_k$  to  $y$  must pass within distance  $8\Delta$  of the vertex  $z$ . Therefore, by the Harnack and Ancona inequalities (24) and (5), for some constant  $C_* = C_R C_{\text{Harnack}}^{32\Delta} < \infty$  independent of  $y, x_k$ ,

$$\begin{aligned} G_R(y, 1) &\leq C_* G_R(y, z) G_R(z, 1) \quad \text{and} \\ G_R(y, x_k) &\leq C_* G_R(y, z) G_R(z, x_k). \end{aligned}$$

On the other hand, by the log-subadditivity of the Green's function,

$$G_R(1, z) G_R(z, x_k) \leq G_R(x_k, 1).$$

It now follows that

$$\begin{aligned} (1 + \delta)^{|y|} G_R(x_k, y) G_R(y, 1) &\leq C_*^2 (1 + \delta)^{|z| + d(z, y)} G_R(z, x_k) G_R(z, y) G_R(y, z) G_R(z, 1) \\ &\leq C_*^2 (1 + \delta)^{|z| + d(z, y)} G_R(x_k, 1) G_R(z, y)^2. \end{aligned}$$

Denote by  $\Gamma(z)$  the set of all vertices  $y \in \Gamma$  such that  $z$  is a closest vertex to  $y$  in the geodesic segment  $L$ . Then for each  $z \in L$ ,

$$\sum_{y \in \Gamma(z)} (1 + \delta)^{d(z, y)} G_R(z, y)^2 \leq \sum_{y \in \Gamma} (1 + \delta)^{|y|} G_R(1, y)^2 = H_1(\delta).$$

Finally, because  $L$  is a geodesic segment from 1 to  $x_k$  there is precisely one vertex  $z \in L$  at distance  $n$  from  $x_k$  for every integer  $0 \leq n \leq |x_k|$ , so  $\sum_{z \in L} (1 + \delta)^{|z|} \leq C_\delta (1 + \delta)^{|x_k|}$  where  $C_\delta = (1 + \delta)/(2 + \delta)$ . Therefore,

$$H_{k+1}(\delta) \leq C_*^2 C_\delta H_1(\delta) H_k(\delta).$$

□

## 7. CRITICAL EXPONENT OF THE GREEN'S FUNCTION AT THE SPECTRAL RADIUS

**7.1. Reduction to Ergodic Theory.** For ease of exposition I will consider only the case  $x = y = 1$  of Theorem 9; the general case can be done in the same manner. The system of differential equations (66) show that the growth of the derivative  $dG_r(1, 1)/dr$  as  $r \rightarrow R-$  is controlled by the growth of the quadratic sums  $\sum_{x \in S_m} G_r(1, x)^2$ . To show that the Green's function has a square root singularity at  $r = R$ , as asserted in (20), it will suffice to show that the (approximate) derivative behaves as follows as  $r \rightarrow R-$ :

**Proposition 46.**

$$(74) \quad \eta(r) := \sum_{x \in \Gamma} G_r(1, x)^2 \sim C/\sqrt{R-r} \quad \text{as } r \rightarrow R-.$$

This will follow from Corollary 48 below. The growth of  $\eta(r)$  as  $r \rightarrow R-$  is related by Propositions 38–41 to that of  $\text{Pressure}(2\varphi_r)$ : in particular, Proposition 41 implies that the dominant contribution to the sum (74) comes from vertices  $x$  at large distances from the root, and so (63) implies that

$$(75) \quad \eta(r) \sim C(R, 2)/(1 - \exp\{\text{Pressure}(2\varphi_r)\})$$

To analyze the behavior of  $\eta(r)$  (or equivalently that of  $\text{Pressure}(2\varphi_r)$ ) as  $r$  approaches the spectral radius, we use the differential equations (66) to express the derivative of  $\eta(r)$  as

$$(76) \quad \frac{d\eta}{dr} = \sum_{x \in \Gamma} \left\{ \sum_{y \in \Gamma} 2r^{-1} G_r(1, x) G_r(1, y) G_r(y, x) \right\} - 2r^{-1} G_r(1, x)^2.$$

(Note: The implicit interchange of  $d/dr$  with an infinite sum is justified here because the Green's functions  $G_r(u, v)$  are defined by power series with nonnegative coefficients.) For  $r \approx R$ , the sum  $\sum_{x \in \Gamma}$  is once again dominated by those vertices  $x$  at large distances from the root 1. The strategy for dealing with the inner sum  $\sum_{y \in \Gamma}$  will be similar to that used in the proof of Lemma 45 above: For each  $x$ , let  $L = L(1, x)$  be the geodesic segment from the root to  $x$  that corresponds to a path in the Cannon automaton, and partition the sum  $\sum_{y \in \Gamma}$  according to the nearest vertex  $z \in L$ :

$$(77) \quad \sum_{y \in \Gamma} = \sum_{z \in L} \sum_{y \in \Gamma(z)}$$

where  $\Gamma(z)$  is the set of all vertices  $y \in \Gamma$  such that  $z$  is a closest vertex to  $y$  in the geodesic segment  $L$ . (If for some  $y$  there are several vertices  $z_1, z_2, \dots$  on  $L$  all closest to  $y$ , put  $y \in \Gamma(z_i)$  only for the vertex  $z_i$  nearest to the root 1.) Log-subadditivity of the Green's function and Ancona's inequalities imply that for  $y \in \Gamma(z)$ ,

$$(78) \quad \begin{aligned} G_r(1, x) G_r(1, y) G_r(y, x) &\asymp G_r(1, z)^2 G_r(z, x)^2 G_r(z, y)^2 \\ &\asymp G_r(1, x)^2 G_r(z, y)^2, \end{aligned}$$

so at least approximately each term in the inner sum in (76) contains a factor  $G_r(1, x)^2$  multiplying the sum  $\sum_{y \in \Gamma(z)} G_r(z, y)^2$ . If this sum were taken over all  $y \in \Gamma$  instead of only those  $y \in \Gamma(z)$ , and if the approximate equalities in (78) were exact, then the inner sum  $\sum_y$  in (76) would reduce to  $(|x| + 1)G_r(1, x)^2\eta(r)$ . Our strategy will be to show, using Corollary 6 and the ergodic theory of the geodesic flow, that for “most” vertices  $x \in S_m$  the *average* value of  $\sum_{v \in \Gamma(z)} G_r(z, y)^2$  along the geodesic segment  $z \in L$  is close to a constant multiple of  $\eta(r)$  for almost every  $x$ :

**Proposition 47.** *For each  $r \leq R$  and each  $m = 1, 2, \dots$  let  $\mu_{r,m}$  be the probability measure on the sphere  $S_m$  with density proportional to  $G_r(1, x)^2$ . There is a continuous, positive function  $\xi(r)$  for  $0 < r \leq R$  such that for each  $r \leq R$  and each  $\varepsilon > 0$ ,*

$$(79) \quad \lim_{m \rightarrow \infty} \mu_{r,m} \left\{ x \in S_m : \left| \frac{1}{m} \sum_{y \in \Gamma} G_r(1, y) G_r(y, x) / G_r(1, x) - \xi(r) \eta(r) \right| > \varepsilon \right\} = 0.$$

The proof is given in section 7.2 below. Given this result, Proposition 46 and Theorem 9 follow easily, as we now show.

**Corollary 48.** *There exists a positive, finite constant  $C$  such that as  $r \rightarrow R-$ ,*

$$(80) \quad \frac{d\eta}{dr} \sim C\eta(r)^3 \quad \text{as } r \rightarrow R-.$$

Consequently,

$$(81) \quad \eta(r)^{-2} \sim C(R - r)/2.$$

*Proof.* The asymptotic relations (78), together with Propositions 38–41 imply that for  $r$  near  $R$ , the dominant contribution to the sum (76) comes from vertices  $x$  far from the root, where Proposition 39 applies. Relation (78) and Proposition 39 imply that the contribution of the second term  $-2r^{-1}G_r(1, x)^2$  to the sum in (76) is asymptotically negligible as  $r \rightarrow R$ , and Proposition 39 (in particular, the fact that  $\xi(r)$  is continuous at  $r = R$ ) implies that

$$\begin{aligned} \sum_{x \in \Gamma} \sum_{y \in \Gamma} 2r^{-1} G_r(1, x) G_r(1, y) G_r(y, x) &\sim 2R^{-1} \xi(R) \eta(r) \sum_{m=1}^{\infty} m \sum_{x \in S_m} G_r(1, x)^2 \\ &\sim C' \eta(r) / (1 - \exp\{\text{Pressure}(2\varphi_r)\})^2 \\ &\sim C \eta(r)^3 \end{aligned}$$

as  $r \rightarrow R-$ , for suitable positive constants  $C, C'$ . This proves (80). The relation (81) follows directly from (80).  $\square$

**7.2. Proof of Proposition 47.** This will be accomplished by showing that the average in (79) can be expressed approximately as an ergodic average of the form (65) covered by Corollary 40. The starting point is the decomposition (77). The inner sum in (77) is over the set  $\Gamma(z)$  of vertices  $y$  for which  $z$  is the nearest point on the geodesic segment  $L$ . The following geometrical lemma implies that the set of relative positions  $z^{-1}y$ , where  $y \in \Gamma(z)$ , depends only on configuration of the geodesic segment  $L$  in a bounded neighborhood of  $z$ .

**Lemma 49.** *If  $L$  and  $L'$  are geodesic segments both passing through the vertex  $z$ , then denote by  $\Gamma(z)$  and  $\Gamma'(z)$ , respectively, the sets of vertices  $y$  such that  $z$  is the nearest<sup>4</sup>*

<sup>4</sup>Assume that the two geodesic segments  $L, L'$  have the same orientation relative to their common segment through  $z$ , so that in cases of multiplicity ties are resolved the same way.

vertex on  $L$  (respectively, on  $L'$ ) to  $y$ . There exists  $K < \infty$ , independent of  $z$ ,  $L$ , and  $L'$ , so that if  $L$  and  $L'$  coincide in the ball of radius  $K$  centered at  $z$ , then

$$(82) \quad \Gamma(z) = \Gamma'(z').$$

*Proof.* This is a routine consequence of the thin triangle property.  $\square$

The next issue is the approximation in (78). For this, the key will be Corollary 6, which will ultimately justify replacing  $G_r(1, y)$  and  $G_r(y, x)$  by the products  $G_r(1, z)G_r(z, y)$  and  $G_r(y, z)G_r(z, x)$ , respectively, times suitable functions of  $z$ . The thin triangle property is essential here, as it implies that, for  $y \in \Gamma(z)$ , any geodesic segments from  $y$  to  $x$  or from  $y$  to 1 must pass within distance  $32\Delta$  of the point  $z$  (see the proof of Lemma 45). Thus, if  $z^+(y)$  and  $z^-(y)$  are the nearest vertices to  $z$  on geodesic segments from  $y$  to  $x$  and  $y$  to 1 (with ties resolved by chronological ordering), then both  $z^+(y)$  and  $z^-(y)$  are among the vertices in the ball of radius  $32\Delta$  centered at  $z$ . The following lemma shows that the assignments  $y \mapsto z^\pm(y)$  can be made so as to depend only on the relative position of  $y$  to  $z$  and the configuration  $(e^+(z), e^-(z))$  of the geodesic  $L(1, x)$  in a bounded neighborhood of  $z$ . (Recall [Corollary 40] that  $e^+(z)$  and  $e^-(z)$  are the sequences of group generators corresponding to the steps of  $L(1, x)$  from  $z$  forward to  $x$  and from  $z$  back to 1, respectively.)

**Lemma 50.** *Assume that the Cayley graph  $G^\Gamma$  is planar. There exists  $K < \infty$  such that the following is true. The assignments  $y \mapsto z^\pm(y)$  for  $y \in \Gamma(z)$  on any geodesic segment  $L(1, x)$  can be made in such a way that the relative positions*

$$z^{-1}z^+(y) \quad \text{and} \quad z^{-1}z^-(y)$$

*depend only on the relative position  $z^{-1}y$  of  $y$  in the sector  $\Gamma(z)$  and the configuration  $(e^+(z), e^-(z))$  of the geodesic  $L(1, x)$  restricted to the ball of radius  $K$  centered at  $z$ .*

*Proof.* The assertion is equivalent to this: For any geodesic segment  $\gamma$  through  $z$  and any vertex  $y \in \Gamma(z)$ , the nearest point  $z_n^+(y)$  to  $z$  on the geodesic segment from  $y$  to a point  $x_n$  on  $\gamma$  outside the ball of radius  $K+1$  centered at  $z$  does not depend on  $x_n$ . If this statement were not true, then for some  $x_n$  and  $x_m$ , the geodesic segments from  $y$  to  $x_n$  and from  $y$  to  $x_m$  would have to cross after their nearest approaches to  $z$ , by planarity of  $G^\Gamma$ . This would contradict the geodesic property for at least one of them.  $\square$

Now consider the terms  $G_r(1, y)G_r(y, x)G_r(1, x)$  in the sum (77). By the Ancona inequalities, the ratios

$$\frac{G_r(1, z^-(y))G_r(z^-(y), y)}{G_r(1, y)}, \quad \frac{G_r(y, z^+(y))G_r(z^+, x)}{G_r(y, x)}, \quad \text{and} \quad \frac{G_r(1, z)G_r(z, x)}{G_r(1, x)}$$

are bounded away from 0 and  $\infty$ , and by Corollary 6 and Lemmas 49–50 they depend continuously on the local configuration  $e^-(z), e^+(z)$  of the geodesic  $L(1, x)$  near  $z$ . By Theorem 5, the ratios

$$\frac{G_r(1, z^-(y))}{G_r(1, z)}, \quad \frac{G_r(z^-(y), y)}{G_r(z, y)}, \quad \frac{G_r(y, z^+(y))}{G_r(z, y)}, \quad \text{and} \quad \frac{G_r(z^+, x)}{G_r(z, x)}$$

also vary continuously with  $e^-(z), e^+(z)$ . Consequently, for a suitable constant  $\xi(r)$ , the convergence (79) follows from Corollary 40. That  $\xi(r)$  varies continuously with  $r$  for  $r \leq R$  follows from the continuous dependence of the Gibbs state  $\mu_r$  with  $r$  (Proposition 39). It remains only to show that  $\xi(R) > 0$ ; this follows from the following lemma.

**Lemma 51.** *There exist  $K < \infty$  and  $C > 0$  independent of  $1 \leq r \leq R$  so that the following is true. For any geodesic segment  $L$  of length  $\geq K$  corresponding to a path in the Cannon automaton, and any  $K$  consecutive vertices  $z_1, z_2, \dots, z_K$  on  $L$ ,*

$$(83) \quad \sum_{j=1}^K \sum_{y \in \Gamma(z_j)} G_r(z, y)^2 \geq C\eta(r).$$

*Proof.* This is in essence a consequence of hyperbolicity, but is easiest to prove using symbolic dynamics. Recall that the sphere  $S_m$  of radius  $m$  in  $\Gamma$  has the description (57) by words of length (approximately)  $m$  in the Cannon automaton  $\mathcal{A}$ . Since the shift  $(\Sigma, \sigma)$  is topologically ergodic and has positive topological entropy, there exists  $K$  so large that any path of length  $\geq K$  has a *fork* in the set of recurrent vertices of  $\mathcal{A}$ , that is, a point where the path could be continued in an alternative fashion. Let  $\gamma$  be the path in the automaton corresponding to  $L$ , and let  $\gamma'$  be a path (possibly much longer than  $\gamma$ ) that agrees with  $\gamma$  up to a fork, where it then deviates from  $\gamma$ . If  $K$  is sufficiently large, then the geodesic  $L'$  corresponding to  $\gamma'$  will be such that for every vertex  $y \in L'$  the nearest vertex to  $y$  in  $L$  will be one of the  $K$  vertices  $z_1, \dots, z_K$ , by Lemma 49. Denote by  $\beta'$  the segment of  $\gamma'$  following the fork from  $\gamma$ . Because the shift  $(\Sigma, \sigma)$  is topologically mixing (Lemma 36), the set of possible continuations  $\beta'$  of length  $m$  nearly coincides with the set of paths  $\beta''$  such that for some short path  $\alpha$  in  $\mathcal{A}$  starting at  $s_*$  the concatenation  $\alpha\beta''$  is a path in  $\mathcal{A}$ . Thus, the sum in (83), which (roughly) corresponds to the sum over all  $\beta'$ , is comparable to the sum over all paths  $\alpha\beta''$  in  $\mathcal{A}$ .  $\square$

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